# Computing, Symbols and Math

*Stephen M. Watt*

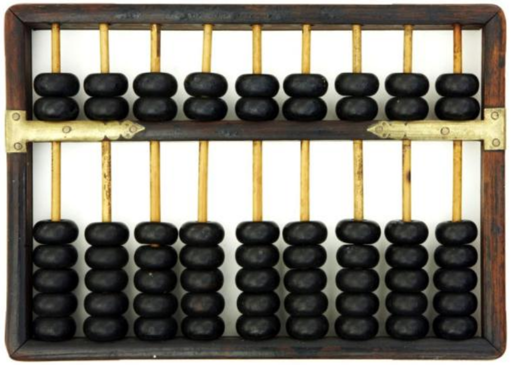
*25 April 2011*

# A Winding Road

- Waterloo
- IBM T.J. Watson Research Centre
- INRIA, University of Nice
- Western

- Mostly computers and mathematics.

# Computing



27 + 15 = 42

6.28318530717959

# Finding an Answer

One day an individual went to the horse races. Instead of counting the number of humans and horses, she counted 74 heads and 196 legs.

How many humans and horses were there?

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$$\text{humans} + \text{horses} = 74$$
$$\text{humans} \times 2 + \text{horses} \times 4 = 196$$



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How many humans and horses were there?

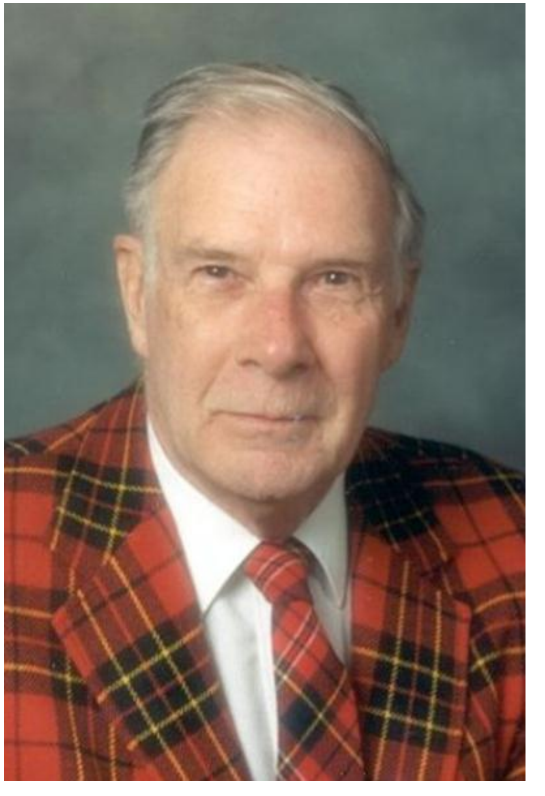
$$\text{humans} + \text{horses} = 74$$
$$\text{humans} \times 2 + \text{horses} \times 4 = 196$$

horses = 24
humans = 50

# What is an Answer?

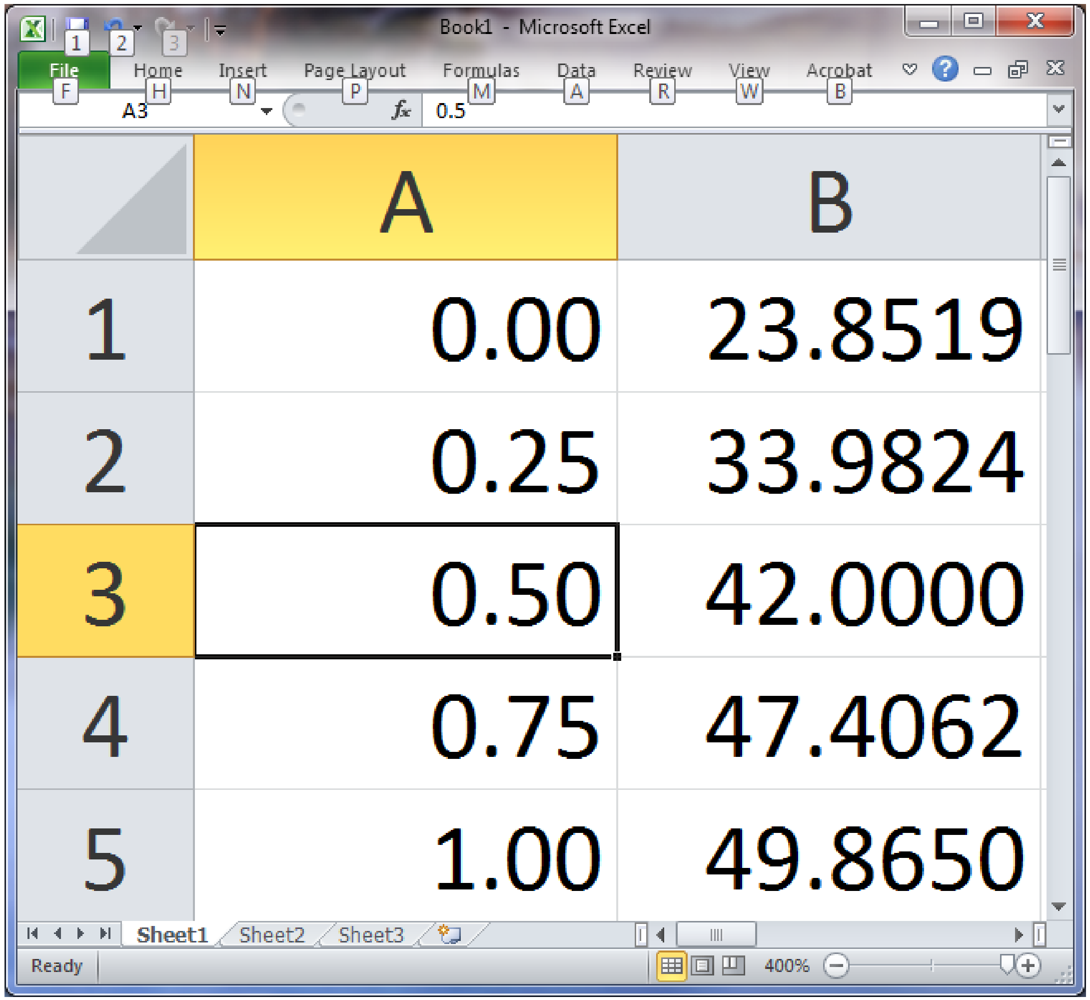
***The purpose of computing is insight, not numbers.***

- Richard Wesley Hamming (1915-1998)
  Dedication to *Introduction to Applied Numerical Analysis* (McGraw Hill 1971)



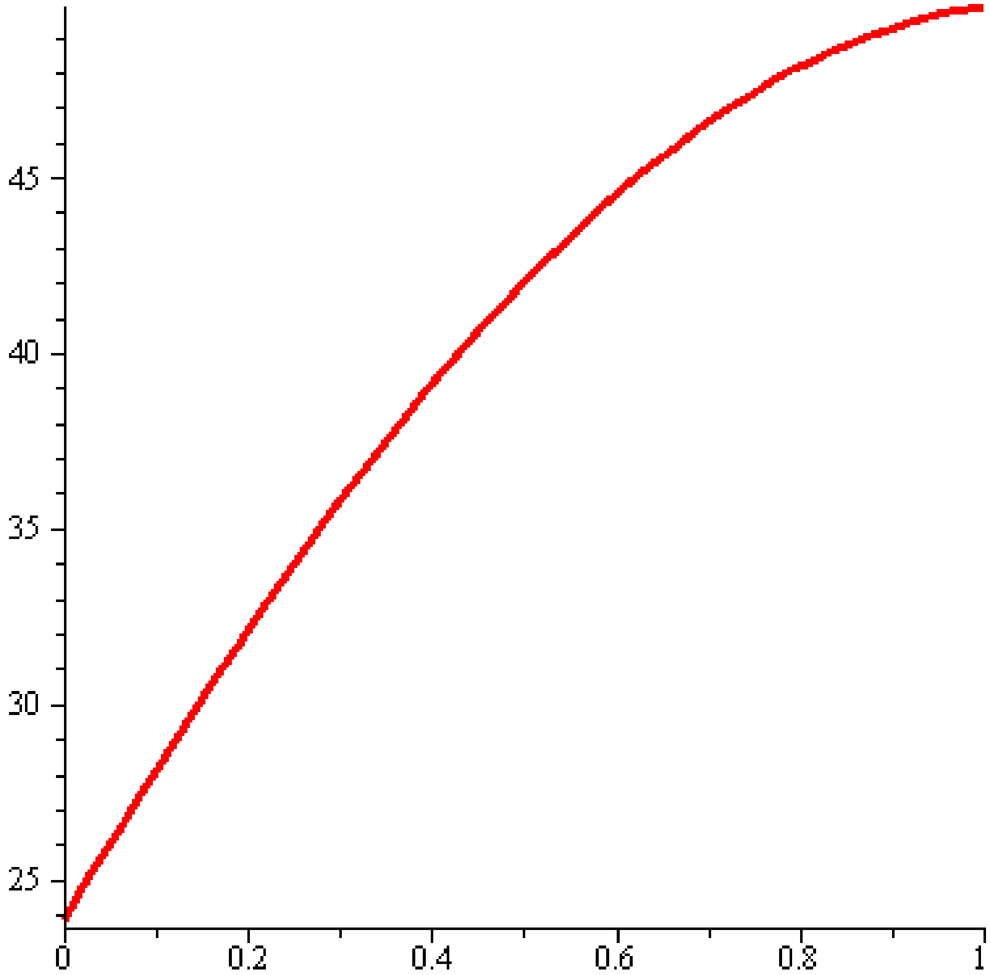
- Hamming codes, Hamming distance, sphere packing
- A founder and president of ACM
- Turing Award winner

# A Better Answer



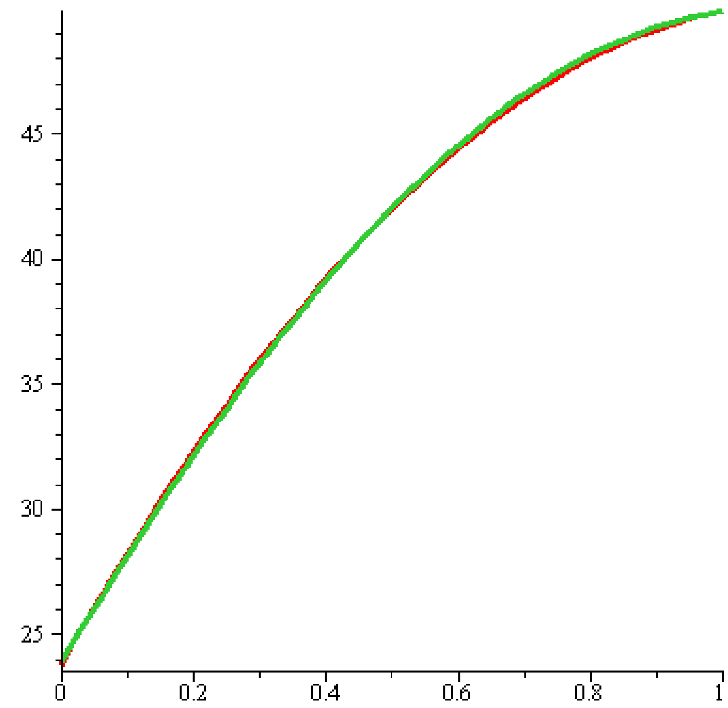
| | A | B |
|---|---|---|
| 1 | 0.00 | 23.8519 |
| 2 | 0.25 | 33.9824 |
| 3 | 0.50 | 42.0000 |
| 4 | 0.75 | 47.4062 |
| 5 | 1.00 | 49.8650 |

# A Better Answer



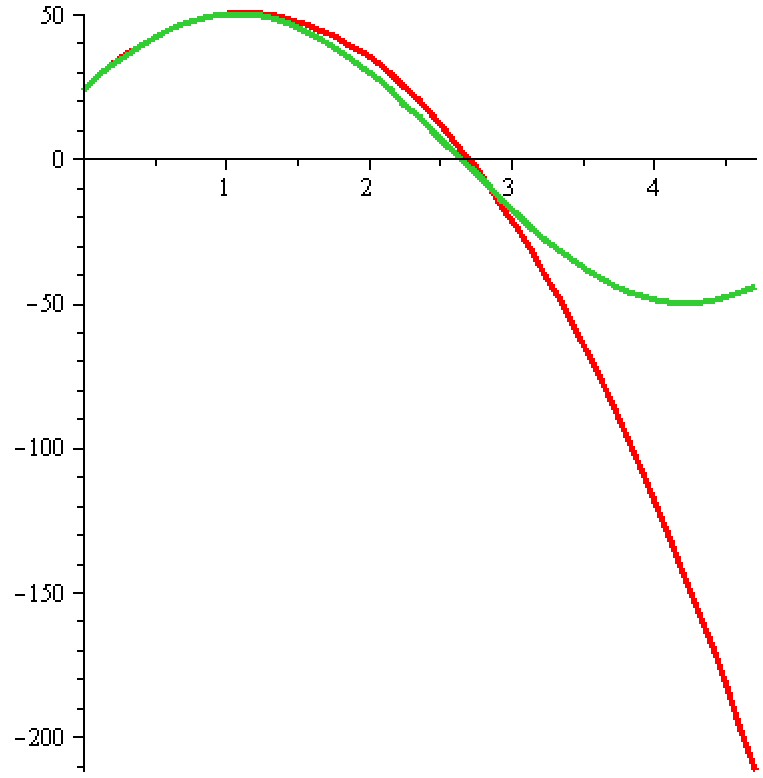
# A Useful Answer

$$-\frac{41}{2}\,x^2+\frac{140}{3}x+\frac{95}{4}$$

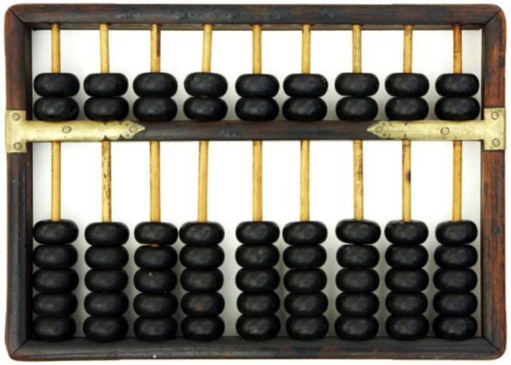


# The Right Answer

$$50\sin\left(x + \frac{1}{2} - \frac{e}{1000}\right)$$



# Symbolic Computing



27 + 15 = 42

6.28318530717959

$$2\pi$$

# Computer Algebra

Having the computer **figure out the formulas**

**rather than** **using formulas** given by humans.

- Algorithms – computational mathematics
- Software – mathematical computation

# Computer Algebra

**Start** with symbols

and **compute** with symbols =>

- **Exact** results
- Hopefully, **insightful** results

# Finding an Answer

One day an individual went to the horse races. Instead of counting the number of humans and horses, she counted $H$ heads and $L$ legs.

How many humans and horses were there?

$$\text{humans} + \text{horses} = H$$
$$\text{humans} \times 2 + \text{horses} \times 4 = L$$

# Finding an Answer

One day an individual went to the horse races. Instead of counting the number of humans and horses, she counted $H$ heads and $L$ legs.

How many humans and horses were there?

$$\begin{aligned} \text{humans} \quad\quad &+ \text{horses} &= \quad H \\ \text{humans} \times 2 \;\; &+ \text{horses} \times 4 &= \quad L \end{aligned}$$

$$\begin{aligned} \text{horses} \;\; &= -H + L/2 \\ \text{humans} &= 2H - L/2 \end{aligned}$$

# Computer Algebra Software

## *Example:* Maple

Untitled (1)* - [Server 1] - Maple 13

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Text Math Drawing Plot Animation Hide

2D Input | Times New Roman | 12

$> 123456789 \cdot 13240123412341234 \cdot 1341341341341341324 1234 \cdot 9999999999999999$

$$2192533918002840700947215483405222354683198679468562525607 29516 \quad (1)$$

$> p := (x^2 + 39 \cdot x + 2) \cdot (x^4 + x^3 - 1) \cdot (x + 1)$

$$p := (x^2 + 39\,x + 2)\,(x^4 + x^3 - 1)\,(x + 1) \quad (2)$$

$> expand(p)$

$$x^7 + 41\,x^6 + 81\,x^5 + x^3 - 40\,x^2 + 43\,x^4 - 41\,x - 2 \quad (3)$$

$> q := sum(x^k, k = 0\,..15)$

$$q := 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} \quad (4)$$

$> factor(q)$

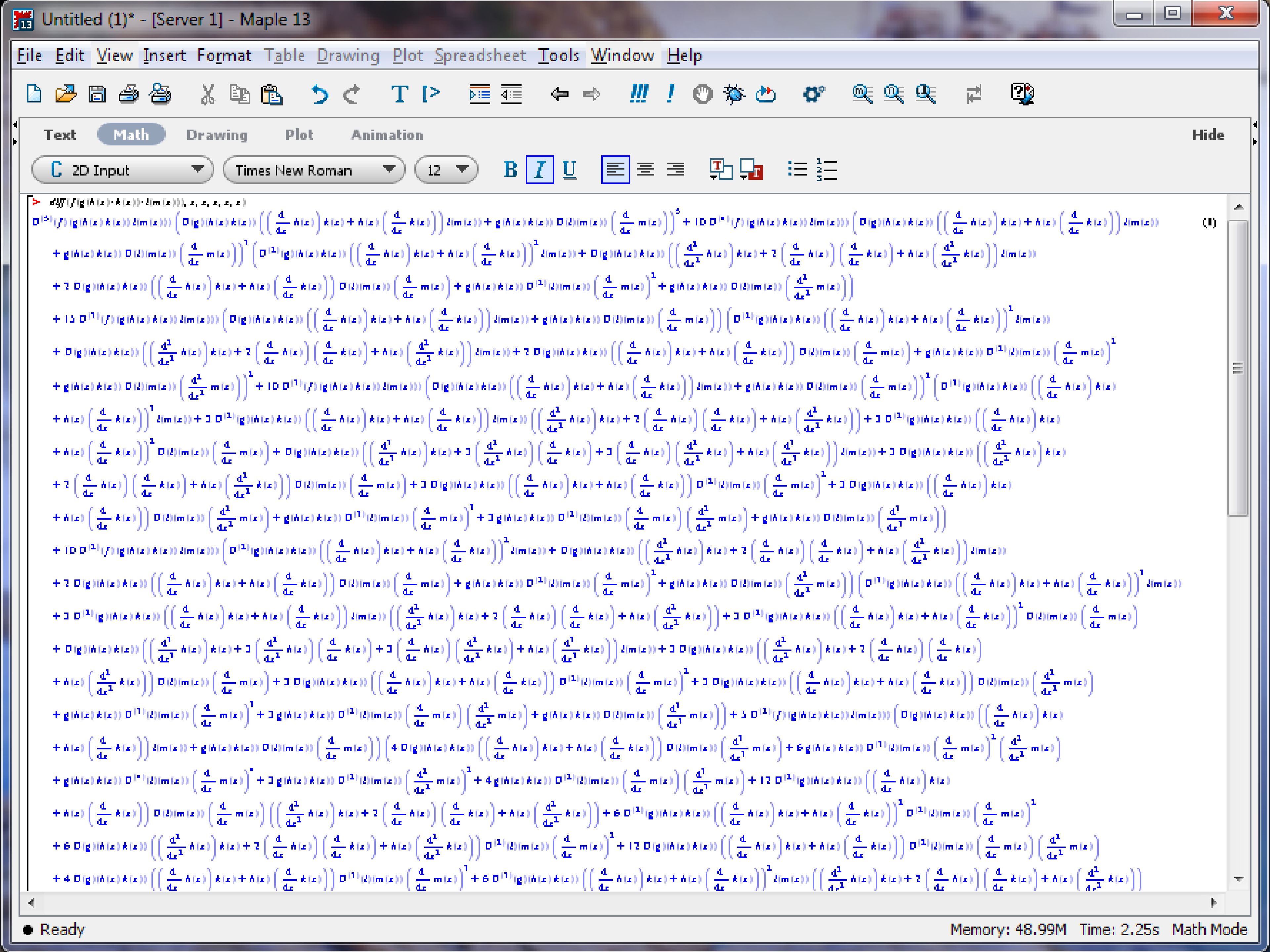
$$(x + 1)\,(1 + x^2)\,(1 + x^4)\,(1 + x^8) \quad (5)$$

$> int\left(\frac{\sin(a \cdot x + b)}{x^2}, x\right)$

$$a\left(-\frac{\sin(a\,x + b)}{a\,x} - \mathrm{Si}(a\,x)\,\sin(b) + \mathrm{Ci}(a\,x)\,\cos(b)\right) \quad (6)$$

$>$

Ready Memory: 0.81M Time: - Math Mode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diff(f(g(h(z)·k(z))·l(m(z))), z, z, z, z, z)
(1)
Ready
Memory: 48.99M Time: 2.25s Math Mode

# Computer Algebra Algorithms

## *Example:* Polynomial Multiplication

- Two polynomials

$$P = 3x^3 + 4x^2 - x + 3 \qquad Q = x^3 - 2x^2 + x + 7$$

- School method

$$\begin{array}{rrrrrrr}
 & & & 3x^3 & +\ 4x^2 & -\ x & +\ 3 \\
 & & \times & x^3 & -\ 2x^2 & +\ x & +\ 7 \\
\hline
 & & & 21x^3 & +28x^2 & -7x & +21 \\
 & & 3x^4 & +\ 4x^3 & -\ \ x^2 & +3x & \\
 & -6x^5 & -8x^4 & +\ 2x^3 & -\ 6x^2 & & \\
3x^6 & +4x^5 & -\ \ x^4 & +\ 3x^3 & & & \\
\hline
3x^6 & -2x^5 & -6x^4 & +30x^3 & +21x^2 & -4x & +21
\end{array}$$

- Multiplication costs *O*(*d* [2] )

# Computer Algebra Algorithms
## *Example:* Polynomial Multiplication

- Point-wise Value Method

Evaluate

$$\begin{aligned} P &= \{(-3,-39),\ (-2,-\ 3),\ (-1,\ 5),\ (0,\ 3),\ (1,\ 9),\ (2,\ 41),\ (3,117)\} \\ Q &= \{(-3,-41),\ (-2,-11),\ (-1,\ 3),\ (0,\ 7),\ (1,\ 7),\ (2,\ \ 9),\ (3,\ 19)\} \end{aligned}$$

$$PQ = \{(-3,1599),\ (-2,\ \ 33),\ (-1,15),\ (0,21),\ (1,63),\ (2,369),\ (3,2223)\}$$

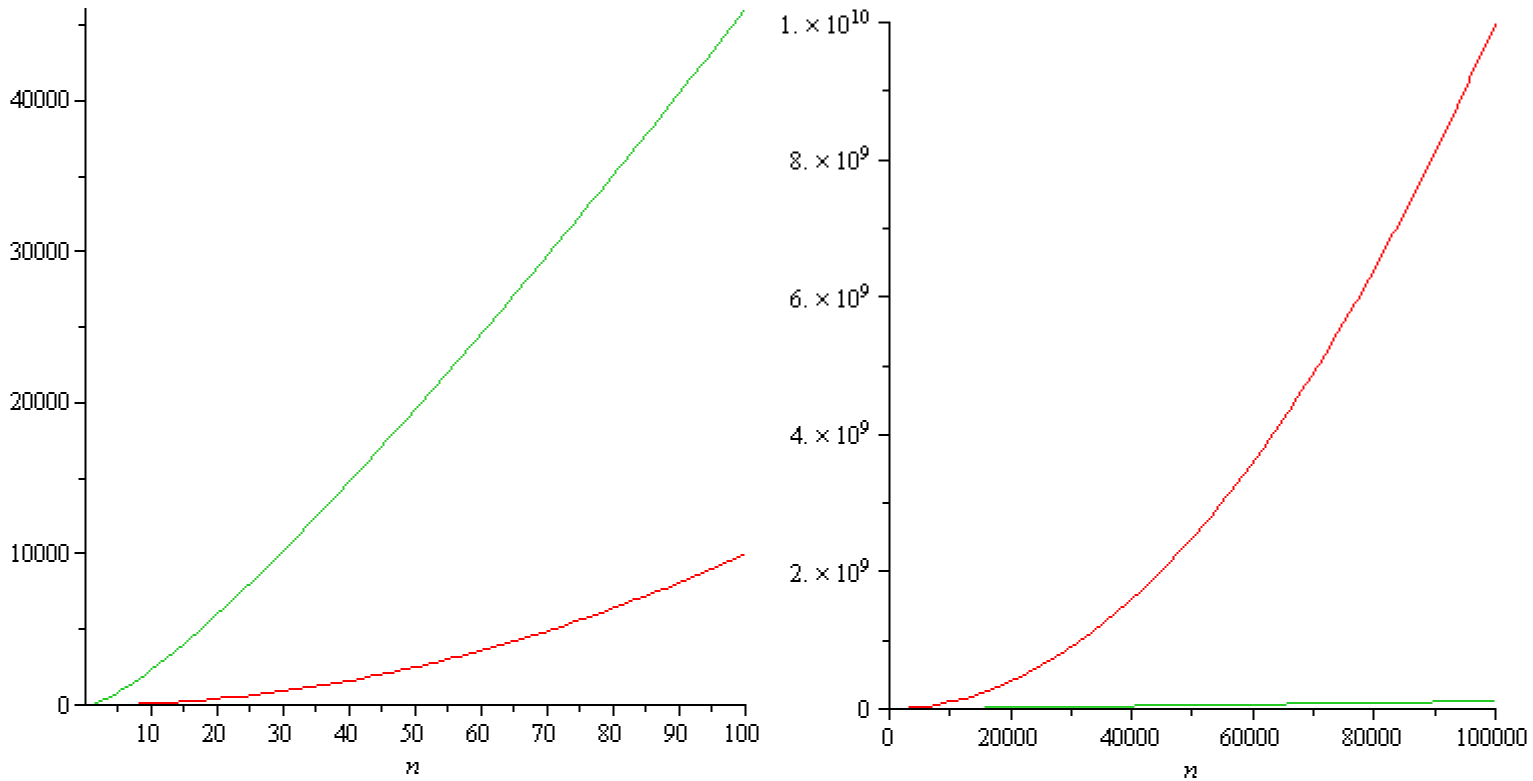
Interpolate

- DFT trick: evaluate at "roots of unity"

$$\omega^0, \omega^1, \omega^2, \ldots \qquad \omega = \sqrt[n]{1} \qquad \text{like} \quad \exp(2\pi i/n) \quad \text{over} \quad \mathbb{C}, \quad \text{but over } \mathbb{F}_p$$

- Multiplication now $O(d \log d)$

# So?


40000
30000
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10000
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10 20 30 40 50 60 70 80 90 100
n
1. × 10^10
8. × 10^9
6. × 10^9
4. × 10^9
2. × 10^9
0
0 20000 40000 60000 80000 100000
n

# For What is Computer Algebra Used?

- Any work involving lengthy formulas, *e.g.*
  - Formulas for location of the moon for Apollo.
  - Martinus Veltman + Gerardus ’t Hooft
    1999 Nobel prize in physics (renormalized YM theory)
  - Cryptographic arms race (making + breaking)

- Education
  - Students can do more interesting examples

- Engineering
  - Faster and more flexible design cycle

# Modeling Canadarm & Canadarm 2

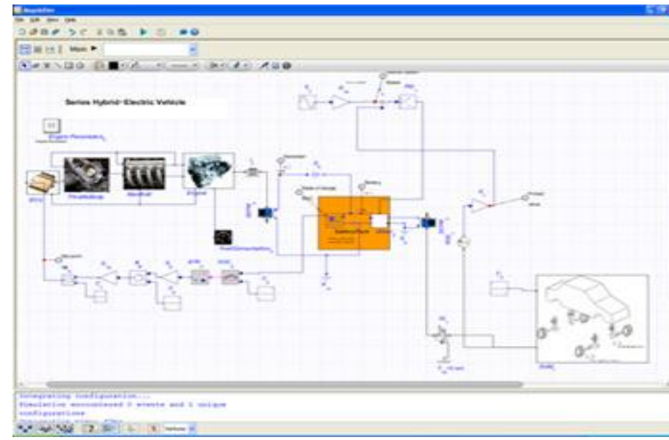
- Each arm has 7 actuators and 22 degrees of freedom.
- Arm cannot support its own weight on Earth so modeling is required.
- Simulink to describe system topology
- Maple manipulates the model and generates C code.


# Computer Animation

- PDI/Dreamworks
- Shading model experiments
- Special FX in volume rendering.
- Partial differential equations for water simulation

# Green Engineering

- High fidelity battery models hybrid-electric vehicles.
- Multi-domain model, automatically generated and optimized equations from MapleSim.

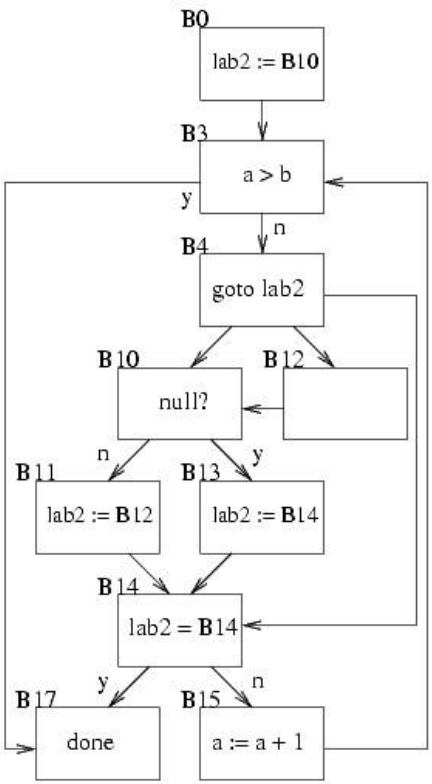



- SkySails GMBH: wind propulsion systems
- Sail + Control System
- Maple used for simulation software.

# Research Interests

- **To expand what mathematical software can do.**

- Compilers and programming languages
- Mathematical algorithms
- Software systems
- Human/computer interfaces



# Algorithms for Symbolic Polynomials

$$p = 8x^{n^2+6n+4+m^2-m} - 2x^{2n^2+7n+2mn}y^{n^2+3n} - 3x^{n^2+3n+2mn}y^{n^2+3n} + 12x^{4+m^2-m+2n}$$

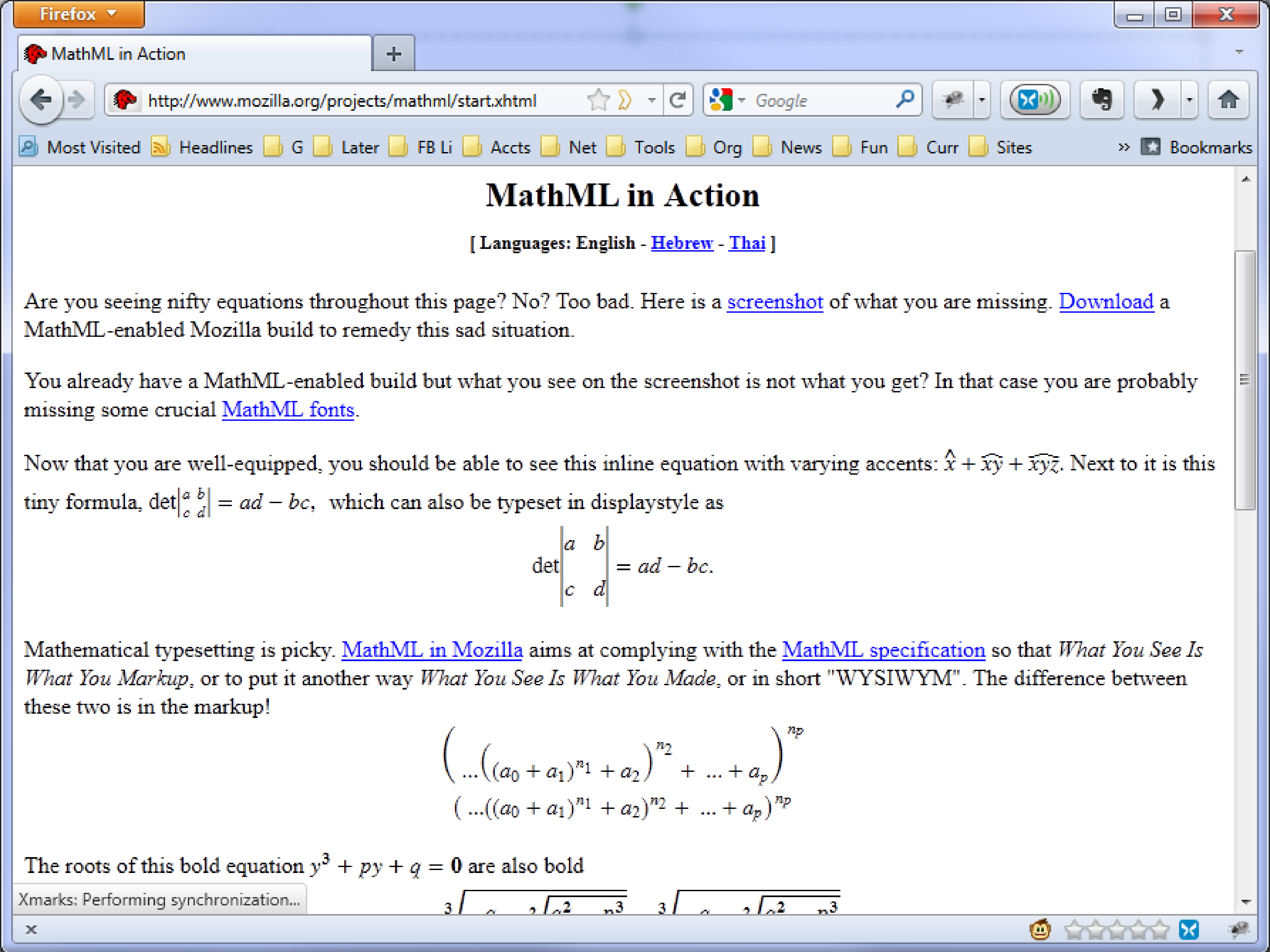
$$= x^{2n} \times \left(2x^{n^2+4n} + 3\right) \times \left(2x^{1/2\,m^2-1/2\,m+2} - x^{1/2\,n^2+mn+1/2\,n}y^{1/2\,n^2+3/2\,n}\right) \times \left(2x^{1/2\,m^2-1/2\,m+2} + x^{1/2\,n^2+mn+1/2\,n}y^{1/2\,n^2+3/2\,n}\right)$$

# Algorithms for Approximate Polynomials

- $f \ = y^2 - x^4 \ \ = \ (y - x^2)(y + x^2)$

- $f^* = y^2 - x^4 + .01x^2$

$$\approx (y \ - x^2 + .00500)(y + x^2 - \ .00504)$$




# MathML in Action

**[ Languages: English - Hebrew - Thai ]**

Are you seeing nifty equations throughout this page? No? Too bad. Here is a screenshot of what you are missing. Download a MathML-enabled Mozilla build to remedy this sad situation.

You already have a MathML-enabled build but what you see on the screenshot is not what you get? In that case you are probably missing some crucial MathML fonts.

Now that you are well-equipped, you should be able to see this inline equation with varying accents: $\hat{x} + \widehat{xy} + \widehat{xyz}$. Next to it is this tiny formula, $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which can also be typeset in displaystyle as

$$\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

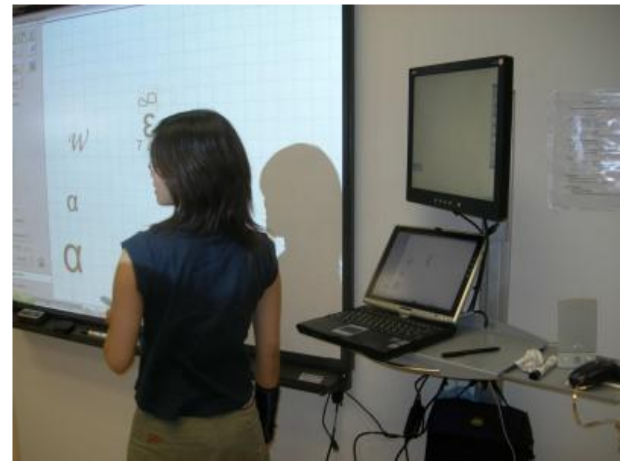
Mathematical typesetting is picky. MathML in Mozilla aims at complying with the MathML specification so that *What You See Is What You Markup*, or to put it another way *What You See Is What You Made*, or in short "WYSIWYM". The difference between these two is in the markup!

$$\left( \ldots \left( (a_0 + a_1)^{n_1} + a_2 \right)^{n_2} + \ldots + a_p \right)^{n_p}$$

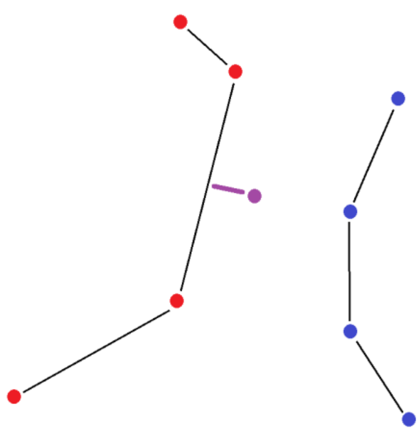
$$( \ldots((a_0 + a_1)^{n_1} + a_2)^{n_2} + \ldots + a_p)^{n_p}$$

The roots of this bold equation $y^3 + py + q = \mathbf{0}$ are also bold

# Math Handwriting Recognition

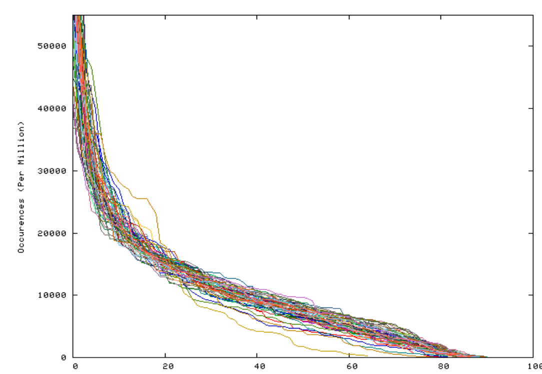


$$e^x = \int e^x dx = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$



$$\sum_{i} i^2$$

$$\dot{z} + z = \sin \omega t$$



# Digital Ink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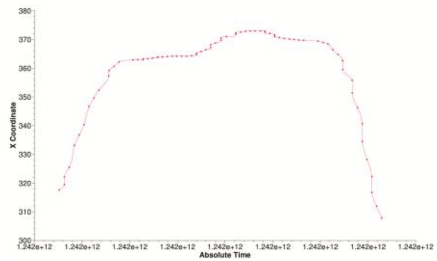
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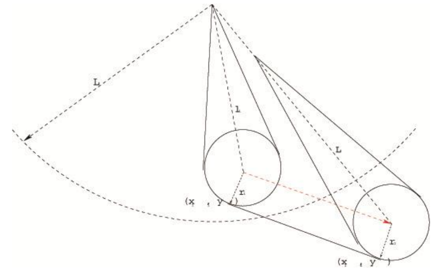
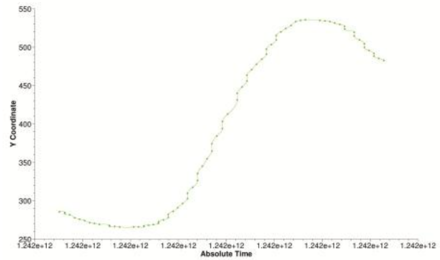
Absolute Time




Y Coordinate
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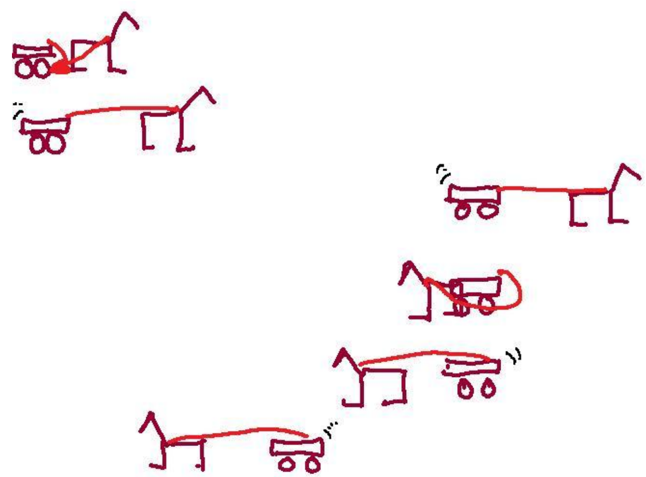

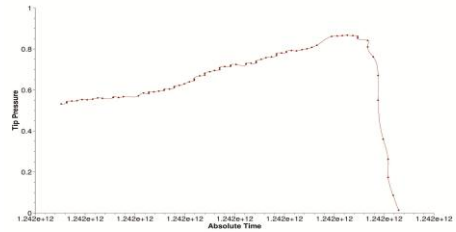
Tip Pressure
1
0.8
0.6
0.4
0.2
0
1.242e+12
Absolute Time

# The Big Picture

- Want computers to be as easy, natural and powerful for mathematics as they are for natural language.

- Powerful programming languages, algorithms, user interfaces – extend these areas as needed.

# Thank you to students and colleagues