Counting problems with parametric polyhedral Sets

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Plan

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Counting memory accesses in a for-loop nest (1/2)

Consider the well-known example SOR (Successive-Over Relaxation) from the numerical solving of PDEs (Partial differential Equations).

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- Using standard techniques from Linear Algebra, namely Fourier-Motzkin elimination (FME), we can rewrite the above set as:

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- Hence the problem becomes counting the number of integer points of a parametric polyhedral set P_N.

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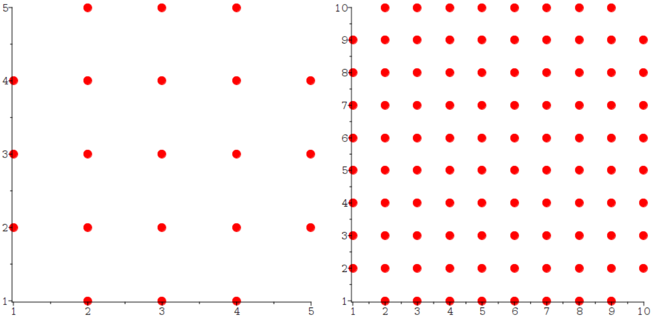

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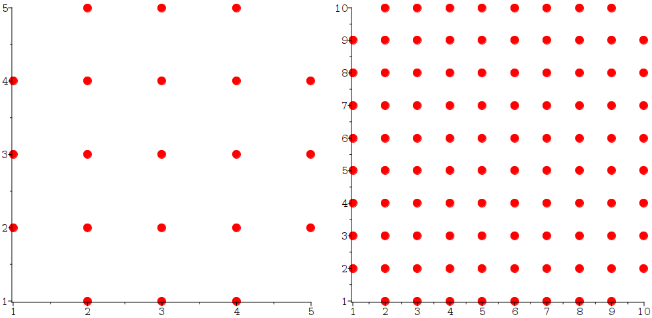

The integer points of the parametric polyhedron P_N for $N = 5$ and $N = 10$.
We will see later that $\mid P \cap \mathbb{Z}^2 \mid \; = \; N^2 - 4$.

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Objective

Our goal is, given a parametric polyhedron $P(\vec{b})$, to count the number of its integer points as a function $c(\vec{b})$ of the parameter $\vec{b}$.

Objective and challenges (2/2)

- One challenge is that the shape (vertices, facets, etc.) of the integer hull of $P(\vec{b})$, that is, $P(\vec{b}) \cap \mathbb{Z}^d$, may vary with the values of $\vec{b}$.

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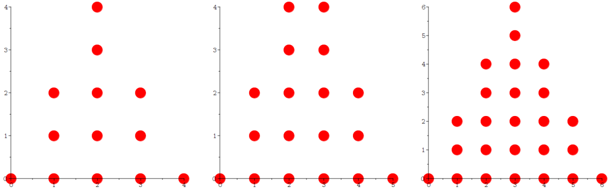

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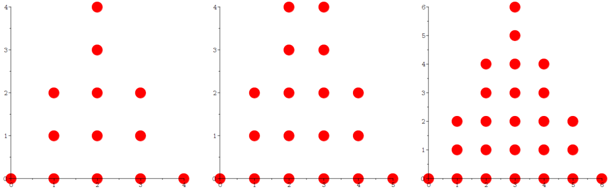

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- Hence, the function $c(\vec{b})$ is computable as a piece-wise function.

Plan

Related works

- Given a 2D polyhedral set (= polytope) P, whose vertices are integer points, Pick's theorem relates the area A of P, the number b of integer points on the border of P, and the number i in the interior of P:

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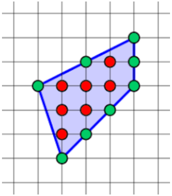

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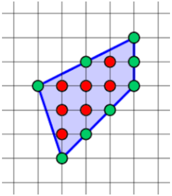

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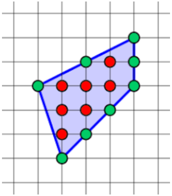

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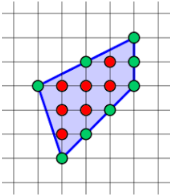

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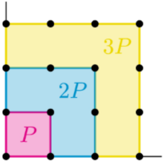

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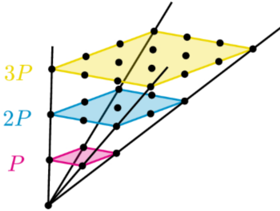

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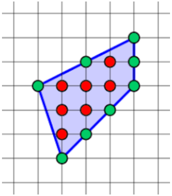

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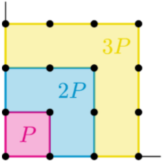
- See Ehrhart polynomial.
- Images are from Wikipedia (fair use category).



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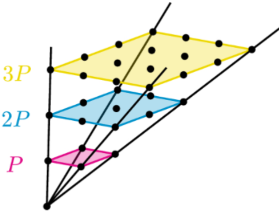

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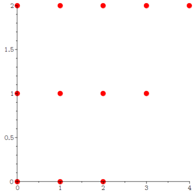
With $d = 2$, we will compute $G(P, \mathbf{x})$ for the polyhedron P given as the convex hull of the 12 points on the figure below.



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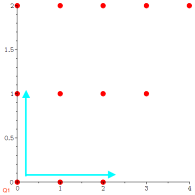

Consider the bottom-left of P, that is, the first quadrant Q_1, that is, the points $(x, , y)$ with $x, y \geq 0$. Then, we have:

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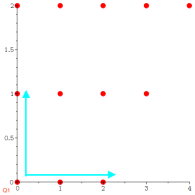

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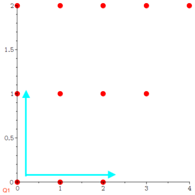

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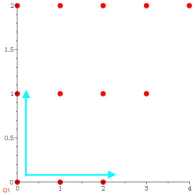

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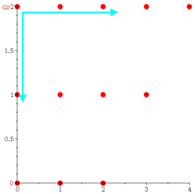

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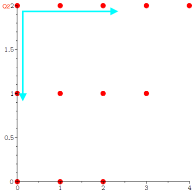

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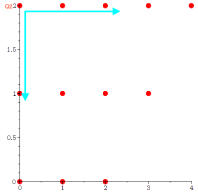

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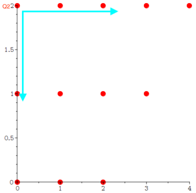

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Continuing with the other corners Q_3 and Q_4 of the polytope P

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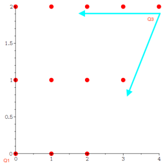

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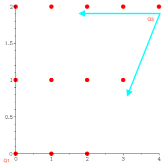

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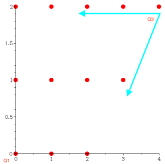

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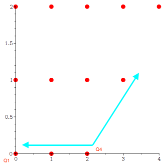

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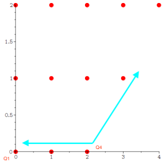

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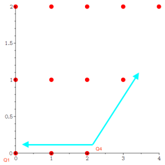

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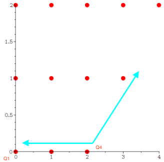

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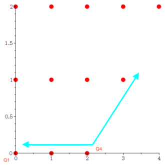

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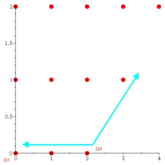

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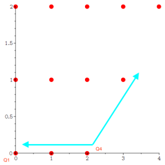

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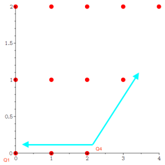

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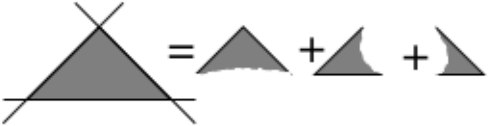
Consequently

$$\begin{array}{rcl} \mid P \cap \mathbb{Z}^2 \mid & = & G(P, (1, 1))) \\ & & = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ & & = 12. \end{array}$$

Plan

Recall Brion's formula

- This formula asserts that for a polytope $P \subseteq \mathbb{Q}^d$ its generating function is the sum of the generating functions of its corners (= vertex cones)



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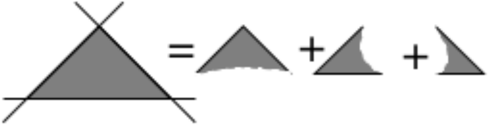

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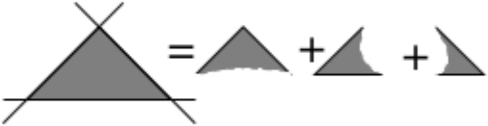

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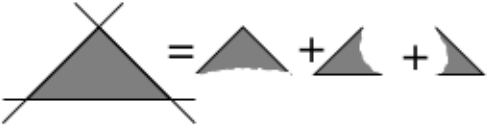

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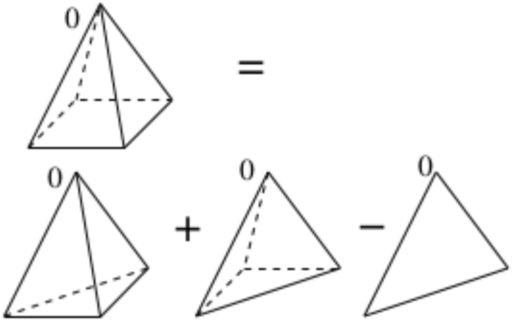
- Our previous calculations used two facts
  1. In dimension $d = 2$, every cone is **simplicial** that is, can be generated by d rays,
  2. The cones $Q2, Q_3, Q_4$ are **unimodular**, that is, the sums of the power series $G(Q_2, \mathbf{x})$, $G(Q_3, \mathbf{x})$, $G(Q_4, \mathbf{x})$ can be deduced from that of $G(Q_1, \mathbf{x})$ (the first quadrant) by means of unimodular transformations (that is, mapping integer vectors to integer vectors).

Barvinok's algorithm

- In dimension d, one can decompose any cone into **simplicial** cones ($=$ cones generated by d rays),



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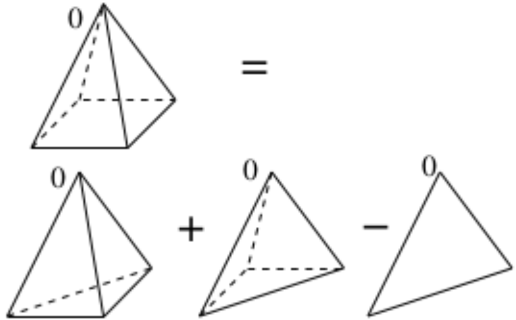

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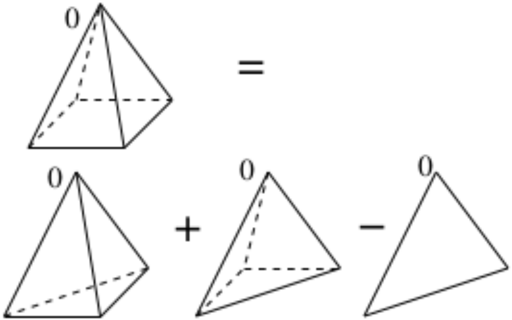

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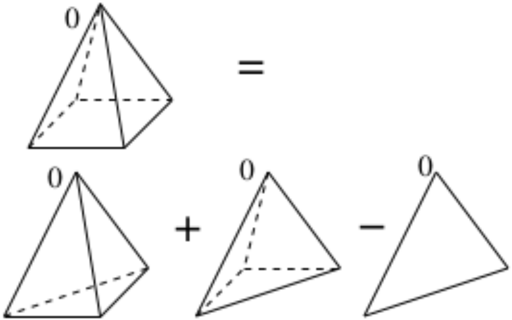

- Alexander Barvinok (1994) proposed an algorithm to decompose any simplicial cones into **unimodular** cones,
- consequently, Barvinok has found the first algorithm to compute $G(P, \mathbf{x})$,
- Moreover, Barvinok's algorithm runs in **polynomial time for a fixed d.**

Plan

Sanity-check examples

Example (1)

Input:

$$\{1 \leq i, 1 \leq j, i \leq n, j \leq n\}$$

Output:

$$[[\{n^2\}, [0 \leq n - 1]]]$$

Sanity-check examples

Example (1)

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Example (3)

Input:

$$\{1 \leq i, 1 \leq j, i+j \leq n, 0 \leq n\}$$

Output:

$$[[\{\frac{n^2}{2} - \frac{n}{2}\}, [0 \leq n-2]]]$$

Examples with several parameters

Example (4)

Input:

$$\{1 \leq i, i \leq n, i \leq m, 1 \leq j, j \leq i\}$$

Output:

$$\begin{gathered}[[\{1\},[m-1=0, 0 \leq n-2]],\\ [\{\tfrac{n^2}{2}+\tfrac{n}{2}\},[0 \leq m-n, 0 \leq n-1]],\\ [\{\tfrac{m^2}{2}+\tfrac{m}{2}\},[0 \leq m-2, 0 \leq n-3, 0 \leq -m+n-1]]]\end{gathered}$$

Examples with several parameters

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Example (5)

Input:

$$\{1 \leq i, i \leq n, i \leq m, 1 \leq j, j \leq p\}$$

Output:

$$\begin{gathered}[[\{pm\},[n-m \geq 1, p-2 \geq 0, m-1 \geq 0]],\\ [\{pn\},[m-n \geq 0, n-2 \geq 0, p-1 \geq 0]],\\ [\{1\},[n-1=0, p-1=0, 0 \leq m-1]],\\ [\{p\},[m-1=0, 0 \leq -2+n, 0 \leq p-1]]]\end{gathered}$$

Examples with quasi-polynomials

Example (6)

Input:

$$\{1 \leq i, j \leq n, 2i \leq 3j\}$$

Output:

$$[[\{Q([n, 2, [\tfrac{3n^2}{4} + \tfrac{n}{2}, -1/4 + \tfrac{3n^2}{4} + \tfrac{n}{2}]])\}, [1 \leq n]]]$$

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Example (7)

Input:

$$\{0 \leq i, 0 \leq j, j \leq 2i, 2i + j \leq n\}$$

Output:

$$\begin{gathered}[[\{Q([n, 4, [1 + \tfrac{n}{2} + \tfrac{n^2}{8}, 3/8 + \tfrac{n}{2} + \tfrac{n^2}{8}, 1/2 + \tfrac{n}{2} + \tfrac{n^2}{8}, 3/8 + \tfrac{n}{2} + \tfrac{n^2}{8}]])\}, \\ [0 \leq n - 1]], \\ [\{1\}, [n = 0]]]\end{gathered}$$

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   4.1 Putting everything together requires computing with **multivariate quasi-polynomials.**

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8. A **value-constraints pair** is any pair (V, C) where $V \subseteq \mathcal{V}$ and C is a system of constraints.

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6. We assume that we have a procedure that, for any system of constraints C, decides whether C is consistent or not.
7. Then, there exists an algorithm that, for the sequence S computes a non-overlapping sequence T refining S.

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   4.4 If for $\gamma : p(\mathbf{x}) \geq 0$ of C_1 we have $p(\mathbf{x}) = -1 - u(\mathbf{x})$ and $u(\mathbf{x}) \geq 0$ is a constraint of C_2, then (p, u) is a pair of **adjacent** inequalities.

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5. Theorem: If (p, u) is a pair of adjacent inequalities, and if all other constraints of C_1 (resp. C_2) are valid on C_2 (resp. C_1) then the system of constraints C_3 consisting of all those valid constraints satisfies $Z(C_3) = Z(C_1) \cup Z(C_2)$.

Plan

Concluding remarks

Summary and notes

1. We have presented Brion's formula and Barvinok's algorithm for computing the number of integer points of a polytope.
2. We have discussed our adaptation of those works to the case of parametric polyhedra and its implementation in MAPLE.
3. Another adaptation to this parametric case, tailored to compiler optimization, was led by Sven Verdoolaege and is part of a C library called `barvinok`.

Work in progress

1. Our MAPLE implementation aims at supporting Presburger arithmetic
2. This implementation is designed to extend to parametric polyhedra $\mathbf{A}\vec{x} \leq \vec{b}$ where parameters appear not only in $\vec{b}$ but also in $\mathbf{A}$.
3. Our current work focuses on minimizing the number of cases in the discussion and controlling expression swell.

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