Polyhedral sets, lattice points, optimizing compilers and computer algebra

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Plan

Acknowledgements

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- Most of the algorithms presented in this tutorial are implemented in MAPLE's `PolyhedralSets` library.

$$
\text{null}\begin{cases}
-x_3 \le 1 \\
-x_1 - x_2 - x_3 \le 2 \\
-x_1 + x_2 - x_3 \le 2 \\
x_1 - x_2 - x_3 \le 2 \\
x_1 + x_2 - x_3 \le 2 \\
x_3 0 \le 1 \\
-x_1 - x_2 + x_3 \le 2 \\
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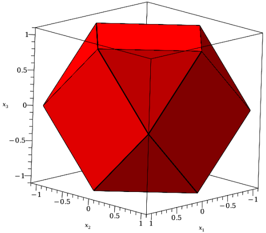
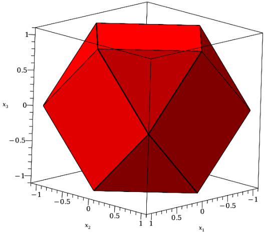
$$



$$\text{null}\begin{cases} -x_3 \leq 1 \\ -x_1 - x_2 - x_3 \leq 2 \\ -x_1 + x_2 - x_3 \leq 2 \\ x_1 - x_2 - x_3 \leq 2 \\ x_1 + x_2 - x_3 \leq 2 \\ x_3 0 \leq 1 \\ -x_1 - x_2 + x_3 \leq 2 \\ -x_1 + x_2 + x_3 \leq 2 \\ x_1 - x_2 + x_3 \leq 2 \\ x_1 + x_2 + x_3 \leq 2 \\ -x_2 0 \leq 1 \\ x_2 \leq 1 \\ -x_1 \leq 1 \\ x_1 0 \leq 1 \end{cases}$$



$$\text{null}\begin{cases} 0 \leq 1 + x_2 \\ 0 \leq 1 - x_2 \\ 0 \leq x_1 + 1 \\ 0 \leq 1 - x_1 \end{cases}$$

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\right.
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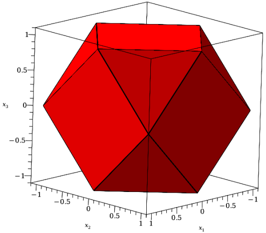

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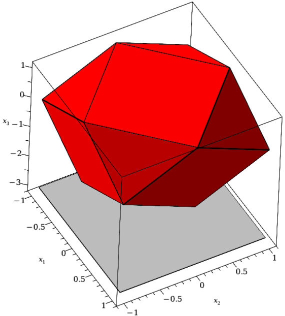

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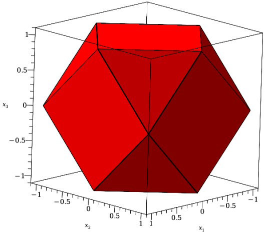

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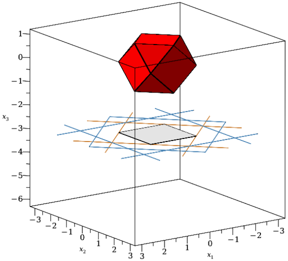

Application of FME: code generation

```
for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i]*b[j];
}
```

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FME reorders $p > t > i > j > n$ to $i > j > t > p > n$, thus eliminating i, j.

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$$\left\{ \begin{aligned} i &= p + t - n \\ j &= -t + n \\ t &\ge \max(0, -p + n) \\ \text{null}\, t &\le \min(n, -p + 2n) \\ 0 &\le p \\ p &\le 2n \\ 0 &\le n. \end{aligned} \right.$$

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The new representation allows us to generate the multithreaded code.

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    c[i+j] += a[i]*b[j];
}
```

```
parallel_for (p=0; p<=2*n; p++){
  c[p] = 0;
  for (t=max(0,n-p);
       t<=min(n,2*n-p);t++)
    c[p] += A[t+p-n] * B[n-t];
}
```

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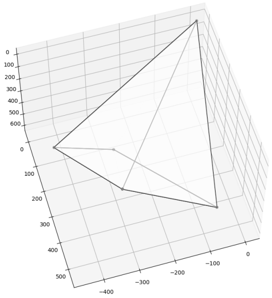
FME reorders $p > t > i > j > n$ to $i > j > t > p > n$, thus eliminating i, j.

skip slide

Application of FME: computing integer hulls (1/3)

The input polyhedral set:

$$\begin{cases} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -5405x_1 + 4965x_2 + 3870x_3 & \leq & 4303504 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{cases}$$



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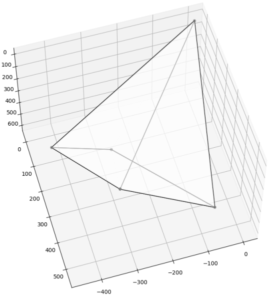
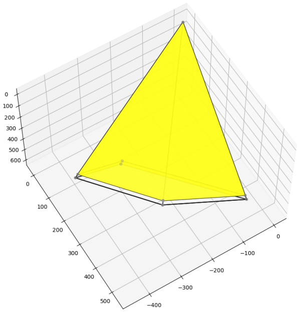
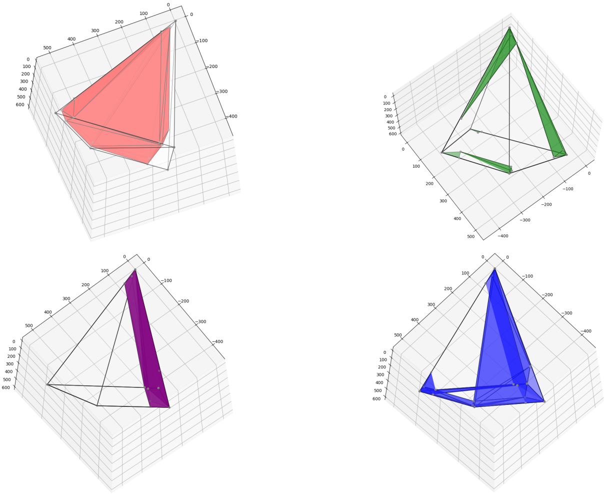


Normalization (leaves the integer hull unchanged):

$$\left\{\begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -1081x_1 + 993x_2 + 774x_3 & \leq & 860700 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$



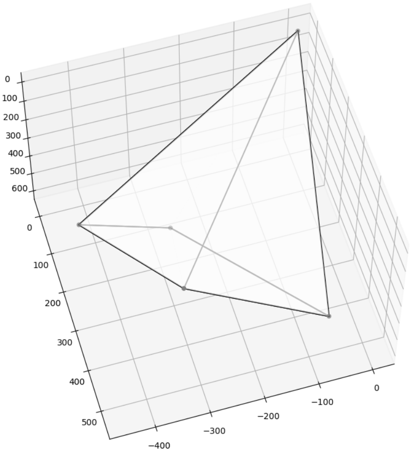
Application of FME: computing integer hulls (2/3)



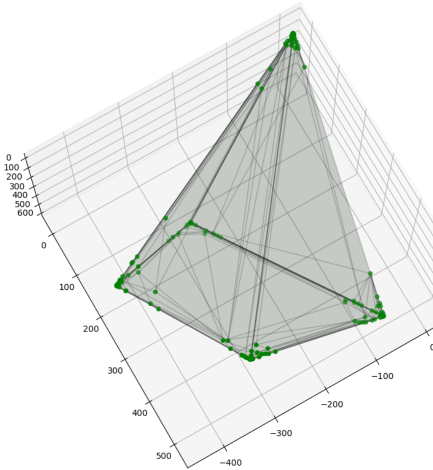
1. The **red** is an approximation of the integer hull of the input.
2. The integer hulls of border regions (green, blue, purple) are brute-force computed via FME.
3. Then `QuickHull` is applied to obtain the integer hull of the input.

Application of FME: computing integer hulls (3/3)

The input has only 5 vertices.



Its integer hull has 139 vertices.



All details are in https://ir.lib.uwo.ca/etd/8985/ and in https://doi.org/10.1007/978-3-031-14788-3_14

Application to dependence analysis

```
for(int i = 0; i < n; i++)
  for(int j = i + 1; j < n; j ++)
    A[i * n + j] = A[(n * j - n + j - i - 1];
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1. Can we parallelize the two `for-loops`?
2. Is there data dependence between two different iterations of the nest?
3. Are there integer solutions to the following system of linear inequalities?

$$
\begin{cases}
\text{null} & \begin{aligned} 0 \leq & i_1 < n \\ i_1 + 1 \leq & j_1 < n \\ 0 \leq & i_2 < n \\ i_2 + 1 \leq & j_2 < n \\ i_1 \times n + j_1 = & n \times j_2 - n + j_2 - i_2 - 1 \end{aligned}
\end{cases}
$$

Delinearize the array accesses

Linearized one-dimensional array

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Delinearized multi-dimensional array

```
for(int i = 0; i < n; i++)
  for(int j = i + 1; j < n; j ++)
    B[i][j] = B[j - 1][j - i - 1];
```

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1. There is no integer solution, therefore, no dependence
2. The problem of delinearization requires to do QE over $\mathbb{Z}$ for non-linear expressions, which is, in principle, unfeasible. But see next slide.

Another approach to the delinearization problem

Principles

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3. unimodularity guarantees that we can map integers to integers back and forth.
4. This can be performed at compile time (at the simple cost of linear algebra) and leads to a case discussion which can be evaluated at execution time.

3D Pattern matching problem

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix} \times \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} ai + bj + ck \\ di + ej + fk \\ gi + hj + lf \end{bmatrix}$$

QE input:

$$\begin{aligned}
&\forall [i_1, j_1, k_1, i_2, j_2, k_2], \\
&(i_1 < i_2) \vee ((i_1 = i_2) \wedge (j_1 < j_2)) \vee ((i_1 = i_2) \wedge (j_1 = j_2) \wedge (k_1 < k_2)) \implies \\
&\quad (a\, i_1 + b\, j_1 + c\, k_1 < a\, i_2 + b\, j_2 + c\, k_2) \\
&\quad \vee((a\, i_1 + b\, j_1 + c\, k_1 = a\, i_2 + b\, j_2 + c\, k_2) \wedge (d\, i_1 + e\, j_1 + f\, k_1 < d\, i_2 + e\, j_2 + f\, k_2)) \\
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&\qquad \wedge(g\, i_1 + h\, j_1 + l\, k_1 < g\, i_2 + h\, j_2 + l\, k_2))
\end{aligned}$$

QE output:

$$(f = 0) \wedge (0 < e) \wedge (c = 0) \wedge (b = 0) \wedge (0 < a) \wedge (0 < l)$$

which gives us the final matrix as below

$$\begin{bmatrix} a > 0 & 0 & 0 \\ c & e > 0 & 0 \\ g & h & l > 0 \end{bmatrix}$$

Plan

1. Overview
2. Basic concepts
3. Solving systems of linear inequalities
4. Integer hulls of polyhedra
5. Integer point counting for parametric polyhedra
6. Quantifier elimination over the integers
7. Concluding remarks

Plan

1. Overview
2. Basic concepts
2.1 Linear, affine, convex and conical hulls
2.2 Polyhedral sets
2.3 Farkas–Minkowsi–Weyl theorem
3. Solving systems of linear inequalities
3.1 Efficient removal of redundant inequalities
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4. Integer hulls of polyhedra
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5.1 Motivations and objectives
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Plan

Affine, conical and convex hulls

Definition 1

Let $X \subseteq \mathbb{R}^d$. The **span**, the **affine hull**, the **convex hull**, the **conical hull** of X, denoted $\mathrm{Span}(X)$, $\mathrm{AffineHull}(X)$, $\mathrm{ConvexHull}(X)$, $\mathrm{ConicalHull}(X)$, are defined by: For every $y \in \mathbb{R}^d$, we have:

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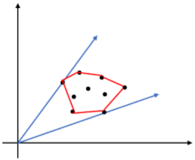
In the plane, the **conical** and **convex** hulls of a few points.



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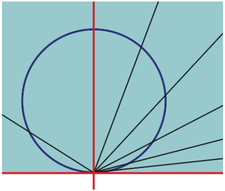
In the plane, the **conical** hull of a **circle** passing through the origin is the open half-plane defined by the tangent line to the circle at the origin plus the origin.



Some properties of hulls and spans (1/3)

Definition 3

For $X, Y \subseteq \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$, we define the **Minkowski sum** of X and Y as

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and we write $X + x_0$ for $X + \{x_0\}$.

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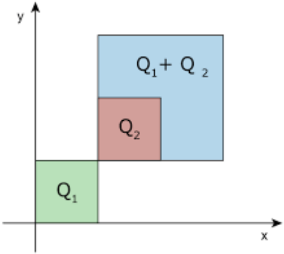

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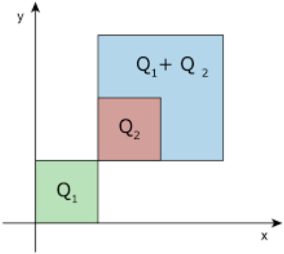

Definition 4

A subset $P \subseteq \mathbb{R}^d$ is a **polytope**, if it is the convex hull of finitely many points, that is, if there exists a finite set $X \subseteq \mathbb{R}^d$ so that $P = \text{ConvexHull}(X)$.

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Proposition 1

If $x_0 \in X$, then we have $\text{AffineHull}(X) = x_0 + \text{Span}(X)$.

Proof.

□

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Assume $x_0 \in X$ and let $x \in \mathrm{AffineHull}(X)$. Then, there exists $(x_1, \ldots, x_e) \in X^e$ and $(\lambda_1, \ldots, \lambda_e) \in \mathbb{R}^e$ such that $x = \lambda_1 x_1 + \cdots + \lambda_e x_e$ and $\lambda_1 + \cdots + \lambda_e = 1$. Then, we have:

$$x = x_0 + \lambda_1(x_1 - x_0) + \cdots \lambda_e(x_e - x_0),$$

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$$x = x_0 + \lambda_1(x_1 - x_0) + \cdots \lambda_e(x_e - x_0),$$

that is, $x \in x_0 +$ Span(X). Conversely, let $x \in\in x_0 +$ Span(X). Let $(\lambda_1, \ldots, \lambda_e) \in \mathbb{R}^e$ such that $x = x_0 + \lambda_1 x_1 + \cdots \lambda_e x_e$. Define $\mu_i = \lambda_i$, for $1 \leq i \leq e$ and $\mu_0 = 1 - \lambda_1 - \cdots - \lambda_e$. Then, we have:

$$x = \mu_0 x_0 + \mu_1 x_1 + \cdots + \mu_e x_e,$$

that is, $x \in$ AffineHull(X). □

Some properties of hulls and spans (2/3)

Proposition 2

We have:

$$\text{ConicalHull}(X) \;=\; \{x \in \mathbb{R}^d \mid \exists t \in \mathbb{R}_{>0}\ tx \in \text{ConvexHull}(X)\} \cup \{0\}.$$

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Proof.

Denote by Z the set on the right-hand side of the equality. Let $x \in Z$. If $x = 0$ holds then $x \in \text{ConicalHull}(X)$ holds. Assume from now on that $x \neq 0$. Let $t \in \mathbb{R}_{>0}$ such that $tx \in \text{ConvexHull}(X)$ holds. Then, let $e \in \mathbb{N}_{>0}$, let $(x_1, \ldots, x_e) \in X^e$, let $(\lambda_1, \ldots, \lambda_e) \in {\mathbb{R}_{\geq 0}}^e$ such that

$$tx = \lambda_1 x_1 + \cdots + \lambda_e x_e \text{ and } \lambda_1 + \cdots + \lambda_e = 1.$$

It follows that $x \in \text{ConicalHull}(X)$ holds.

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It follows that $x \in \text{ConicalHull}(X)$ holds. Conversely, let $x \in \text{ConicalHull}(X)$. If $x = 0$ holds then $x \in Z$ holds. Assume from now on that $x \neq 0$. Then, let $e \in \mathbb{N}_{>0}$, let $(x_1, \ldots, x_e) \in X^e$, let $(\lambda_1, \ldots, \lambda_e) \in {\mathbb{R}_{\geq 0}}^e$ such that

$$x = \lambda_1 x_1 + \cdots + \lambda_e x_e \text{ holds.}$$

Since $x \neq 0$, we have $\lambda := \lambda_1 + \cdots + \lambda_e \neq 0$. It follows that $\frac{x}{\lambda} \in X$, that is, $x \in Z$. □

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3. $\text{ConvexHull}(X)$ *contains any convex set that contains* X*.*

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4. $X \subseteq \mathrm{ConicalHull}(X)$ *holds and* $\mathrm{ConicalHull}(X)$ *is convex.*

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Supporting hyperplanes

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Let $\alpha \in \mathbb{R}^d$*, let* $\beta \in \mathbb{R}$ *and denote by* H *the hyperplane defined by*

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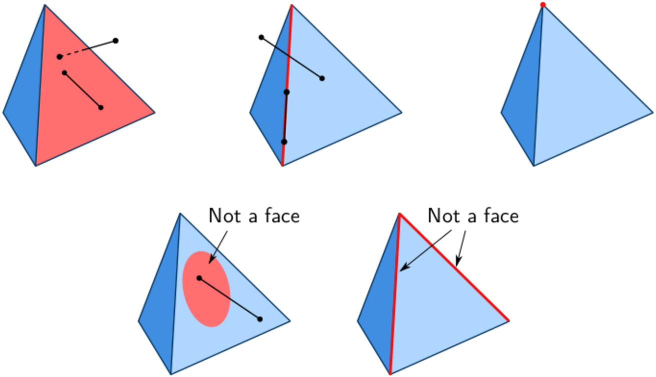

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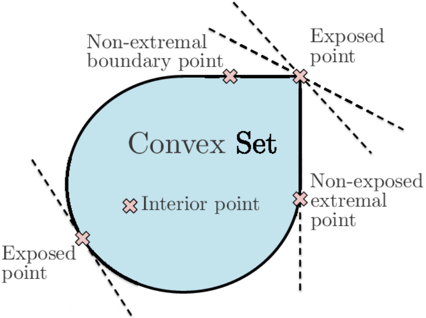

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7. *Assume* K *is closed. let* $L = \{\lambda z \mid \lambda \geq 0\}$ *be a ray emanating from the origin and let* $x, y \in K$. *Then, we have:*
$$x + L \subseteq K \iff y + L \subseteq K.$$

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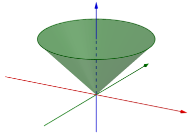
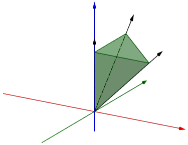


A non-polyhedral cone.



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For a subset $C \subseteq \mathbb{R}^d$, the **dual cone** is given by:

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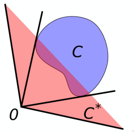
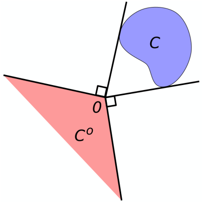


A cranberry and its dual.



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Polar cone theorem

Theorem 12

For a subset $C \subseteq \mathbb{R}^d$, we have:

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Plan

Polyhedra

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1. A subset $P \subseteq \mathbb{R}^d$ is a **convex polyhedron** (or simply a polyhedron) if

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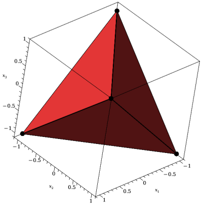

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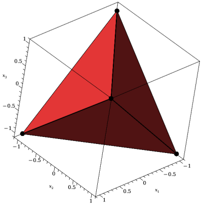
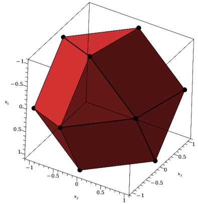
$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$$
holds, for a matrix $A \in \mathbb{R}^{m \times d}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, where $m \in \mathbb{N}_{>0}$.
2. we call the linear system $\{A\mathbf{x} \leq \mathbf{b}\}$ an **H-representation** of P and denote by Polyhedron$(A, \mathbf{b})$ the polyhedron P, that is, the solution set of the system of linear inequalities $A\mathbf{x} \leq \mathbf{b}$.

Remark 1

1. *A polyhedron is the intersection of finitely many affine half-spaces.*
2. *Therefore, a polyhedron is both closed and convex.*





Implicit or redundant equalities (1/3)

Notations 4

1. *Let again* $P := \text{Polyhedron}(A, \mathbf{b})$.

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2. *Let* $\mathbf{c} \in \mathbb{R}^d$ *and* $\beta \in \mathbb{R}$ *such that the inequality* $\mathbf{c}^t\mathbf{x} \leq \beta$ *is one of the inequalities of* $A\mathbf{x} \leq \mathbf{b}$.

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1. We say that $\mathbf{c}^t\mathbf{x} \le \beta$ is an **redundant inequality** of $A\mathbf{x} \le \mathbf{b}$ if it is implied by the other inequalities defining P.

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$$A\mathbf{x} \leq \mathbf{b} \implies \mathbf{c}^t\mathbf{x} = \beta.$$

Implicit or redundant equalities (1/3)

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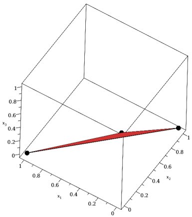

Implicit or redundant equalities (2/3)

Notations 5

Following [18], we denote by $A^{=}$ (resp. A^{+}) and $\mathbf{b}^{=}$ (resp. $\mathbf{b}^{+}$) the rows of A and $\mathbf{b}$ corresponding to the implicit (resp. non-implicit) equalities.

Implicit or redundant equalities (2/3)

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Following [18], we denote by $A^=$ (resp. A^+) and $\mathbf{b}^=$ (resp. $\mathbf{b}^+$) the rows of A and $\mathbf{b}$ corresponding to the implicit (resp. non-implicit) equalities.

Proposition 6

If P is not empty, then there exists $\mathbf{x} \in P$ satisfying both

$$A^= \mathbf{x} = \mathbf{b}^= \quad \text{and} \quad A^+ \mathbf{x} < \mathbf{b}^+. \tag{2.1}$$

Implicit or redundant equalities (2/3)

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Following [18], we denote by $A^{=}$ *(resp.* A^{+}*) and* $\mathbf{b}^{=}$ *(resp.* $\mathbf{b}^{+}$*) the rows of* A *and* $\mathbf{b}$ *corresponding to the implicit (resp. non-implicit) equalities.*

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If P *is not empty, then there exists* $\mathbf{x} \in P$ *satisfying both*

$$A^{=}\mathbf{x} \ = \ \mathbf{b}^{=} \ \text{ and } \ A^{+}\mathbf{x} \ < \ \mathbf{b}^{+}. \qquad (2.1)$$

Proof.

Assume P is not empty and has at least one non-implicit equality. Denote by $\mathbf{c}_1^t\mathbf{x} \leq \beta_1, \ldots, \mathbf{c}_e^t\mathbf{x} \leq \beta_e$. For each $1 \leq i \leq e$, there exists $\mathbf{x}_i \in P$ so that we have $\mathbf{c}_e^t\mathbf{x}_i \leq \beta_e$. Define:

$$\mathbf{x} \ = \ \frac{\mathbf{x}_1 + \cdots + \mathbf{x}_e}{e}.$$

Then $\mathbf{x}$ satisfies Equation 2.1 null. □

Implicit or redundant equalities (3/3)

Proposition 7

If Polyhedron$(A, \mathbf{b})$ *is not empty, then we have:*

$$\begin{array}{rcl} \text{AffineHull}(\text{Polyhedron}(A, \mathbf{b})) & = & \{\mathbf{x} \in \mathbb{R}^d \mid A^{=}\mathbf{x} = \mathbf{b}^{=}\} \\ & = & \{\mathbf{x} \in \mathbb{R}^d \mid A^{=}\mathbf{x} \leq \mathbf{b}^{=}\}. \end{array}$$

Implicit or redundant equalities (3/3)

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Proof.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_e \in$ Polyhedron$(A, \mathbf{b})$, let $\lambda_1, \ldots, \lambda_e \in \mathbb{R}$, and let $\mathbf{c}^t\mathbf{x} \leq \beta$ be an implicit equality. Since $\mathbf{c}^t\mathbf{x}_i = \beta$ holds for all $1 \leq i \leq e$, we have:

$$\mathbf{c}^t \left(\Sigma_{i=1}^{e} \lambda_i \mathbf{x}_i\right) = \beta.$$

The inclusion $\subseteq$ follows.

□

Implicit or redundant equalities (3/3)

Proposition 7

If Polyhedron$(A, \mathbf{b})$ *is not empty, then we have:*

$$\text{AffineHull}(\text{Polyhedron}(A, \mathbf{b})) = \{\mathbf{x} \in \mathbb{R}^d \mid A^= \mathbf{x} = \mathbf{b}^=\} = \{\mathbf{x} \in \mathbb{R}^d \mid A^= \mathbf{x} \le \mathbf{b}^=\}.$$

Proof.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_e \in$ Polyhedron$(A, \mathbf{b})$, let $\lambda_1, \ldots, \lambda_e \in \mathbb{R}$, and let $\mathbf{c}^t \mathbf{x} \le \beta$ be an implicit equality. Since $\mathbf{c}^t \mathbf{x}_i = \beta$ holds for all $1 \le i \le e$, we have:

$$\mathbf{c}^t \left(\Sigma_{i=1}^{e} \lambda_i \mathbf{x}_i \right) = \beta.$$

The inclusion $\subseteq$ follows. Conversely, let $\mathbf{x}_0$ satisfy $A^= \mathbf{x} \le \mathbf{b}^=$. Let $\mathbf{x}_1 \in$ Polyhedron$(A, \mathbf{b})$. Using Proposition 6 null, we can assume that $A^+ \mathbf{x}_1 < \mathbf{b}^+$ holds. If $\mathbf{x}_0 = \mathbf{x}_1$, then we have $\mathbf{x}_0 \in$ Polyhedron$(A, \mathbf{b})$ and thus $\mathbf{x}_0 \in$ AffineHull(Polyhedron$(A, \mathbf{b})$). Otherwise, using a continuity argument, the segment joining $\mathbf{x}_0$ and $\mathbf{x}_1$ contains more than one point in Polyhedron$(A, \mathbf{b})$, and thus more than one point in AffineHull(Polyhedron$(A, \mathbf{b})$). Therefore, the whole segment $[\mathbf{x}_0, \mathbf{x}_1]$ is necessarily contained in that latter set, which proves the inclusion $\supseteq$. □

The faces of a polyhedral set (1/2)

Notations 6

Let again $P := \text{Polyhedron}(A, \mathbf{b})$ *be a polyhedron of* $\mathbb{R}^d$.

Proposition 8

A non-empty subset $F \subseteq P$ *is a* **face** *of* P *if* $F = \{\mathbf{x} \in P | A'\mathbf{x} = \mathbf{b}'\}$ *for some subsystem* $A'\mathbf{x} \leq \mathbf{b}'$ *of* $A\mathbf{x} \leq \mathbf{b}$.

Definition 15

1. A face of P, distinct from P, and with maximum dimension is a **facet** of P.
2. The **lineality space** of P is $\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} = \vec{0}\}$ and P is said **pointed** if its lineality space has dimension zero.
3. For a pointed polyhedron P, the inclusion-minimal faces are the **vertices** of P, that is, its 0-dimensional faces.

The faces of a polyhedral set (2/2)

Theorem 16

If Polyhedron$(A, \mathbf{b})$ *is full-dimensional (that is, has dimension d) and the the system $A\mathbf{x} \leq \mathbf{b}$ has no redundant inequalities, then the inequalities of that system are in 1-to-1 correspondence with the facets of P.*

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If Polyhedron$(A, \mathbf{b})$ is full-dimensional (that is, has dimension d) and the the system $A\mathbf{x} \leq \mathbf{b}$ has no redundant inequalities, then the inequalities of that system are in 1-to-1 correspondence with the facets of P.

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1. *The faces of P, when ordered by the set theoretic inclusion, form a lattice L which enjoys three important properties (that are not true in general for an arbitrary lattice):*

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   - c. *if the ranks of two faces $a > b$ differ by 2, then there are exactly 2 faces that lie strictly between a and b.*
2. *the face lattice of a polytope can be uniquely determined from its facets, its vertices and its vertex-facet incidences.*

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Linear programming (review)

Notations 7

1. *Let* $A \in \mathbb{R}^{m \times d}$ *be a matrix, let* $\mathbf{c} \in \mathbb{R}^d$ *and* $\mathbf{b} \in \mathbb{R}^m$ *be two vectors*

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1. *Weak duality: If both programs have feasible solutions, then*

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Remark 2

The same results hold if the constraints are $A\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}$ and $A^t\mathbf{y} \le \mathbf{c}$, respectively.

Fundamental theorem of linear inequalities

Theorem 18

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{b} \in \mathbb{R}^d$*, Denote by* r *the rank of the* $d \times (m+1)$ *matrix whose columns are* $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{b}$*. Then exactly one of the following statements holds:*

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2. *There exists a hyperplane* $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{c}^t \mathbf{x} = 0\}$ *containing* $r-1$ *linearly independent vectors from* $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ *such that we have:*

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1. *There exists s linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and s numbers $\lambda_1, \ldots, \lambda_s \in \mathbb{R}_{\geq 0}$ so that $b = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s$,*
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    - a. $\mathbf{c}^t \mathbf{b} < 0$,
    - b. $\mathbf{c}^t \mathbf{a}_i \geq 0$, *for all $1 \leq i \leq m$.*

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2. *There exists a hyperplane* $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{c}^t \mathbf{x} = 0\}$ *containing* $r - 1$ *linearly independent vectors from* $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ *such that we have:*
   a. $\mathbf{c}^t \mathbf{b} < 0$,
   b. $\mathbf{c}^t \mathbf{a}_i \geq 0$*, for all* $1 \leq i \leq m$*.*

   Moreover, if $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *are rational then* $\mathbf{c}$ *is rational as well.*

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Proof.

1. A constructive proof based on the simplex algorithm in Schrijver's book [18].
2. For a shorter proof, one can weaken the above statement and proving Gordan's theorem, instead.
3. The proof of Gordan's theorem is very similar to that of the bipolar theorem.

Farkas lemma

Theorem 19

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{b} \in \mathbb{R}^d$*, Denote by* A *the* $m \times d$ *matrix whose columns are* $\mathbf{a}_1, \ldots, \mathbf{a}_m$*. Then exactly one of the following statements holds:*

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2. *there exists* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^t A \geq 0, \mathbf{y}^t b < 0.$

Farkas lemma

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Let $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{b} \in \mathbb{R}^d$*, Denote by* A *the* $m \times d$ *matrix whose columns are* $\mathbf{a}_1, \ldots, \mathbf{a}_m$*. Then exactly one of the following statements holds:*

1. *the system* $A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}$ *has solutions,*
2. *there exists* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^t A \geq 0, \mathbf{y}^t b < 0$.

Proof.

We apply Theorem 18 null.

□

Farkas lemma

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Proof.

We apply Theorem 18 null. Suppose that here exists s linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and s numbers $\lambda_1, \ldots, \lambda_s \in \mathbb{R}_{\geq 0}$ so that $b = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s$. This is equivalent to say that the system $A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$ has solutions.

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Suppose now that there exists a hyperplane $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{c}^t\mathbf{x} = 0\}$ containing $r - 1$ linearly independent vectors from $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ such that we have $\mathbf{c}^t\mathbf{b} < 0$ and $\mathbf{c}^t\mathbf{a}_i \geq 0$, for all $1 \leq i \leq m$. Set $\mathbf{y} = \mathbf{c}$. Then, this implies that both $\mathbf{y}^t A \geq 0$ and $\mathbf{y}^t b < 0$ hold. □

Plan

Farkas–Minkowsi–Weyl theorem (1/4)

Theorem 20

A convex cone $C \subseteq \mathbb{R}^d$ is polyhedral if and only if it is finitely generated.

Farkas–Minkowsi–Weyl theorem (1/4)

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Assume there exists $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ so that $C = \text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

□

Farkas–Minkowsi–Weyl theorem (1/4)

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Assume there exists $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ so that $C = \text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

1. W.l.o.g. assume that $\text{Span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \mathbb{R}^d$, otherwise do the proof in $\text{Span}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$. Thus we have $d \leq m$.

□

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2. By the fundamental theorem of linear inequalities, the following statements are equivalent, for all $\mathbf{y} \in \mathbb{R}^d$:
   - a. $\mathbf{y} \in \text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$,
   - b. for every hyperplane $\{\mathbf{x} \mid \mathbf{c}^t\mathbf{x} = 0\}$ containing $d-1$ linearly independent vectors from $\mathbf{x}_1, \ldots, \mathbf{x}_m$ so that $\mathbf{c}^t\mathbf{x}_{\geq}0$ holds for all $1 \leq i \leq m$, we have $\mathbf{c}^t\mathbf{y} \geq 0$.

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3. Consider all hyperplanes spanned by $d - 1$ linearly independent vectors from $\mathbf{x}_1, \ldots, \mathbf{x}_m$ so that $\mathbf{c}^t\mathbf{x}_i \geq 0$ holds for all $1 \leq i \leq m$.

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4. The number of such hyperplanes is at most $N := \binom{m}{d-1}$.

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4. The number of such hyperplanes is at most $N := \binom{m}{d-1}$.
5. Let those hyperplanes be defined by $\mathbf{c}_1, \ldots, \mathbf{c}_N$.

□

Farkas–Minkowsi–Weyl theorem (2/4)

Proof.

We shall prove that we have:

$$\text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \; = \; \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x}\mathbf{c}_i^t \geq 0, 1 \leq i \leq N\}, \qquad (2.2)$$

which will imply that the cone C is polyhedral. We prove the two inclusions $\subseteq$ and $\supseteq$.

□

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- $\subseteq$: This inclusion follows immediately from our above observation derived from the fundamental theorem of linear inequalities.

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which will imply that the cone C is polyhedral. We prove the two inclusions $\subseteq$ and $\supseteq$.

- $\subseteq$: This inclusion follows immediately from our above observation derived from the fundamental theorem of linear inequalities.
- $\supseteq$: Consider $\mathbf{y} \in \mathbb{R}^d$ so that $\mathbf{y} \notin \text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$. From the same observation, there exists an hyperplane H_i among those defined by $\mathbf{c}_1, \ldots, \mathbf{c}_N$ such that $\mathbf{x}\mathbf{c}_i^t \geq 0$ does **not** hold.

□

Farkas–Minkowsi–Weyl theorem (2/4)

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We shall prove that we have:

$$\text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \; = \; \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x}\mathbf{c}_i^t \geq 0, 1 \leq i \leq N\}, \tag{2.2}$$

which will imply that the cone C is polyhedral. We prove the two inclusions $\subseteq$ and $\supseteq$.

- $\subseteq$: This inclusion follows immediately from our above observation derived from the fundamental theorem of linear inequalities.
- $\supseteq$: Consider $\mathbf{y} \in \mathbb{R}^d$ so that $\mathbf{y} \notin \text{ConicalHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$. From the same observation, there exists an hyperplane H_i among those defined by $\mathbf{c}_1, \ldots, \mathbf{c}_N$ such that $\mathbf{x}\mathbf{c}_i^t \geq 0$ does **not** hold.

Assume now that exist $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^d$ so that we have

$$C = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t\mathbf{x} \leq \mathbf{0}, \; 1 \leq i \leq m\}, \tag{2.3}$$

that is, assume C is polyhedral. □

Farkas–Minkowsi–Weyl theorem (3/4)

Proof.

1. From the first part of the proof, there exist vectors $\mathbf{b}_1, \ldots, \mathbf{b}_N \in \mathbb{R}^d$ such that we have:

$$\text{ConicalHull}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{b}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq N\}, \quad (2.4)$$

□

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2. Note that for all $1 \leq j \leq m$ and all $1 \leq i \leq N$, we have:
$$\mathbf{b}_i^t \mathbf{a}_i \leq 0, \quad (2.5)$$
since $\mathbf{a}_j, \in$ ConicalHull$(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ holds.

□

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3. We will show that C is a finitely generated, by proving:
$$C \;=\; \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N). \quad (2.6)$$

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Farkas–Minkowsi–Weyl theorem (3/4)

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5. Observe that $\mathbf{b}_i \in C$, since $\mathbf{b}_i^t \mathbf{a}_i \le 0$ holds for all $1 \le j \le m$ and all $1 \le i \le N$.

□

Farkas–Minkowsi–Weyl theorem (3/4)

Proof.

1. From the first part of the proof, there exist vectors $\mathbf{b}_1, \ldots, \mathbf{b}_N \in \mathbb{R}^d$ such that we have:
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2. Note that for all $1 \leq j \leq m$ and all $1 \leq i \leq N$, we have:
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since $\mathbf{a}_j, \in$ ConicalHull$(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ holds.
3. We will show that C is a finitely generated, by proving:
$$C = \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N). \quad (2.6)$$
4. We prove ConicalHull$(\mathbf{b}_1, \ldots, \mathbf{b}_N) \subseteq \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\}$.
5. Observe that $\mathbf{b}_i \in C$, since $\mathbf{b}_i^t \mathbf{a}_i \leq 0$ holds for all $1 \leq j \leq m$ and all $1 \leq i \leq N$.
6. It follows that all linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_N$ with non-negative coefficients are in C. This proves the inclusion.

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

1. So let $\mathbf{y} \in C$ and $\mathbf{y} \notin \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$.

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq$ ConicalHull$(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

1. So let $\mathbf{y} \in C$ and $\mathbf{y} \notin$ ConicalHull$(\mathbf{b}_1, \ldots, \mathbf{b}_N)$.
2. From the first part of the proof, there exist vectors $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{R}^d$ such that we have:

$$\text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{w}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq r\}, \quad (2.7)$$

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

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3. Consequently, there exists some $1 \leq i \leq r$, such that $\mathbf{w}_i^t \mathbf{y} > 0$.

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t\mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \ \subseteq\ \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

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3. Consequently, there exists some $1 \leq i \leq r$, such that $\mathbf{w}_i^t\mathbf{y} > 0$.
4. By definition of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ we have: $\mathbf{w}_i^t\mathbf{b}_j \leq 0$, for all $1 \leq j \leq N$.

□

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$$\text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{w}_i^t\mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq r\}, \quad (2.7)$$
3. Consequently, there exists some $1 \leq i \leq r$, such that $\mathbf{w}_i^t\mathbf{y} > 0$.
4. By definition of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ we have: $\mathbf{w}_i^t\mathbf{b}_j \leq 0$, for all $1 \leq j \leq N$.
5. Since $\text{ConicalHull}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{b}_i^t\mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq N\}$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$ such that we have: $\mathbf{w}_i^t = \Sigma_{k=1}^m \lambda_k \mathbf{a}_k$.

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

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$$\text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{w}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq r\}, \quad (2.7)$$
3. Consequently, there exists some $1 \leq i \leq r$, such that $\mathbf{w}_i^t \mathbf{y} > 0$.
4. By definition of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ we have: $\mathbf{w}_i^t \mathbf{b}_j \leq 0$, for all $1 \leq j \leq N$.
5. Since $\text{ConicalHull}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{b}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq N\}$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$ such that we have: $\mathbf{w}_i^t = \sum_{k=1}^m \lambda_k \mathbf{a}_k$.
6. Since $\mathbf{y} \in C$ and $C = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\}$, we deduce
$$\mathbf{w}_i^t \mathbf{y} = \sum_{k=1}^m \lambda_k \mathbf{a}_k^t \mathbf{w}_i \leq 0. \quad (2.8)$$

□

Farkas–Minkowsi–Weyl theorem (4/4)

Proof.

We prove $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\} \subseteq \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$, by contradiction.

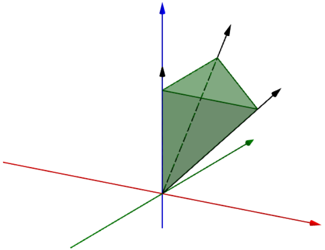
1. So let $\mathbf{y} \in C$ and $\mathbf{y} \notin \text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N)$.
2. From the first part of the proof, there exist vectors $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{R}^d$ such that we have:
$$\text{ConicalHull}(\mathbf{b}_1, \ldots, \mathbf{b}_N) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{w}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq r\}, \quad (2.7)$$
3. Consequently, there exists some $1 \leq i \leq r$, such that $\mathbf{w}_i^t \mathbf{y} > 0$.
4. By definition of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ we have: $\mathbf{w}_i^t \mathbf{b}_j \leq 0$, for all $1 \leq j \leq N$.
5. Since $\text{ConicalHull}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{b}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq N\}$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$ such that we have: $\mathbf{w}_i^t = \sum_{k=1}^m \lambda_k \mathbf{a}_k$.
6. Since $\mathbf{y} \in C$ and $C = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}_i^t \mathbf{x} \leq \mathbf{0},\ 1 \leq i \leq m\}$, we deduce
$$\mathbf{w}_i^t \mathbf{y} = \sum_{k=1}^m \lambda_k \mathbf{a}_k^t \mathbf{w}_i \leq 0. \quad (2.8)$$
7. In contradiction with $\mathbf{w}_i^t \mathbf{y} > 0$.

□

Decomposition theorem for polyhedron (1/6)

Theorem 21

Let $P \subseteq \mathbb{R}^d$. Then, P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some polyhedral cone C.



Proof.

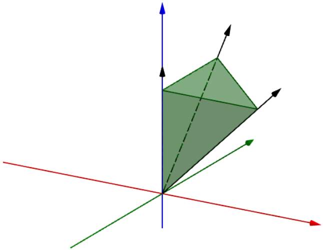
1. We will show that it follows from Farkas-Minkowsi-Weyl theorem.

□

Decomposition theorem for polyhedron (1/6)

Theorem 21

Let $P \subseteq \mathbb{R}^d$. Then, P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some polyhedral cone C.



Proof.

1. We will show that it follows from Farkas-Minkowsi-Weyl theorem.
2. For this, we will rely on the geometric intuition that a polyhedron is a slice of a cone.

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron.

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron.
2. We are looking for a polytope Q and a cone C such that $P = Q + C$.

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron.
2. We are looking for a polytope Q and a cone C such that $P = Q + C$.
   a. Define:

$$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda \mathbf{b} \leq \mathbf{0}, \lambda \geq 0 \right\}.$$

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \le \mathbf{b}\}$ be a polyhedron.
2. We are looking for a polytope Q and a cone C such that $P = Q + C$.
   - a. Define:
   $$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda\mathbf{b} \le \mathbf{0}, \lambda \ge 0 \right\}.$$
   - b. Note that T is a polyhedral cone.

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \le \mathbf{b}\}$ be a polyhedron.
2. We are looking for a polytope Q and a cone C such that $P = Q + C$.
   a. Define:
   $$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda\mathbf{b} \le \mathbf{0}, \lambda \ge 0 \right\}.$$
   b. Note that T is a polyhedral cone.
   c. From the Farkas-Minkowsi-Weyl theorem, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such hat
   $$T = \text{ConicalHull}(\begin{pmatrix} \mathbf{x}_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{x}_m \\ \lambda_m \end{pmatrix}).$$

□

Decomposition theorem for polyhedron (2/6)

Proof.

1. Let $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron.
2. We are looking for a polytope Q and a cone C such that $P = Q + C$.
   a. Define:
   $$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda\mathbf{b} \leq \mathbf{0}, \lambda \geq 0 \right\}.$$
   b. Note that T is a polyhedral cone.
   c. From the Farkas-Minkowsi-Weyl theorem, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such hat
   $$T = \text{ConicalHull}(\begin{pmatrix} \mathbf{x}_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{x}_m \\ \lambda_m \end{pmatrix}).$$
   d. By rescaling the elements of T, we can assume $\lambda_i \in \{0, 1\}$, for all i.

□

Decomposition theorem for polyhedron (3/6)

Proof.

Recall

$$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda\mathbf{b} \leq \mathbf{0}, \lambda \geq 0 \right\} = \text{ConicalHull}(\begin{pmatrix} \mathbf{x}_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{x}_m \\ \lambda_m \end{pmatrix}),$$

with $\lambda_i \in \{0, 1\}$, for all i.

□

Decomposition theorem for polyhedron (3/6)

Proof.

Recall

$$T = \left\{ \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} - \lambda \mathbf{b} \leq \mathbf{0}, \lambda \geq 0 \right\} = \text{ConicalHull}\left(\begin{pmatrix} \mathbf{x}_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{x}_m \\ \lambda_m \end{pmatrix} \right),$$

with $\lambda_i \in \{0, 1\}$, for all i. For all $\mathbf{y} \in \mathbb{R}^d$ we have:

$$\begin{array}{rcl} \mathbf{y} \in P & \Longleftrightarrow & (\exists \lambda \in \mathbb{R}_{\geq 0}) \begin{pmatrix} \mathbf{y} \\ \lambda \end{pmatrix} \in T \\ & \Longleftrightarrow & (\exists \lambda, \gamma_1, \ldots, \gamma_m \in \mathbb{R}_{\geq 0}) \begin{pmatrix} \mathbf{y} \\ \lambda \end{pmatrix} = \gamma_1 \begin{pmatrix} \mathbf{x}_1 \\ \lambda_1 \end{pmatrix} + \cdots + \gamma_m \begin{pmatrix} \mathbf{x}_m \\ \lambda_m \end{pmatrix} \\ & \Longleftrightarrow & (\exists \gamma_1, \ldots, \gamma_m \in \mathbb{R}_{\geq 0}) \left\{ \begin{array}{l} \mathbf{y} = \sum_{i=1}^m \gamma_i \mathbf{x}_i \\ \sum_{i=1, \lambda_i = 1}^m \gamma_i = 1 \end{array} \right. \\ & \Longleftrightarrow & (\exists \gamma_1, \ldots, \gamma_m \in \mathbb{R}_{\geq 0}) \left\{ \begin{array}{l} \mathbf{y} = \sum_{i=1, \lambda_i = 0}^m \gamma_i \mathbf{x}_i + \sum_{i=1, \lambda_i = 1}^m \gamma_i \mathbf{x}_i \\ \sum_{i=1, \lambda_i = 1}^m \gamma_i = 1 \end{array} \right. \\ & \Longleftrightarrow & (\exists \gamma_1, \ldots, \gamma_m \in \mathbb{R}_{\geq 0}) \ \mathbf{y} \in C + Q, \end{array}$$

where:

$$C = \text{ConicalHull}(\mathbf{x}_i \mid \lambda_i = 0) \quad \text{and} \quad Q = \text{ConvexHull}(\mathbf{x}_i \mid \lambda_i = 1).$$

□

Decomposition theorem for polyhedron (4/6)

Proof.

1. Let $P = C + Q$, where C is a polyhedral cone and Q is a polytope.

□

Decomposition theorem for polyhedron (4/6)

Proof.

1. Let $P = C + Q$, where C is a polyhedral cone and Q is a polytope.
2. We need to show that P is a polyhedron.

□

Decomposition theorem for polyhedron (4/6)

Proof.

1. Let $P = C + Q$, where C is a polyhedral cone and Q is a polytope.
2. We need to show that P is a polyhedron.
3. From the Farkas-Minkowsi-Weyl theorem, there exist $\mathbf{r}_1, \ldots, \mathbf{r}_t \in \mathbb{R}^d$ so that $C = \text{ConicalHull}(\mathbf{r}_1, \ldots, \mathbf{r}_t)$.

□

Decomposition theorem for polyhedron (4/6)

Proof.

1. Let $P = C + Q$, where C is a polyhedral cone and Q is a polytope.
2. We need to show that P is a polyhedron.
3. From the Farkas-Minkowsi-Weyl theorem, there exist $\mathbf{r}_1, \ldots, \mathbf{r}_t \in \mathbb{R}^d$ so that $C = \text{ConicalHull}(\mathbf{r}_1, \ldots, \mathbf{r}_t)$.
4. By definition of a polytope, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ so that $Q = \text{ConvexHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

□

Decomposition theorem for polyhedron (4/6)

Proof.

1. Let $P = C + Q$, where C is a polyhedral cone and Q is a polytope.
2. We need to show that P is a polyhedron.
3. From the Farkas-Minkowsi-Weyl theorem, there exist $\mathbf{r}_1, \ldots, \mathbf{r}_t \in \mathbb{R}^d$ so that $C = \text{ConicalHull}(\mathbf{r}_1, \ldots, \mathbf{r}_t)$.
4. By definition of a polytope, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^d$ so that $Q = \text{ConvexHull}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.
5. Then, we have:

$$\mathbf{y} \in P \iff (\exists \lambda_i, \gamma_j \in \mathbb{R}_{\geq 0}) \begin{cases} \mathbf{y} = \sum_{i=1}^{t} \lambda_i \mathbf{r}_i + \sum_{j=1}^{m} \gamma_j \mathbf{x}_j \\ \sum_{j=1}^{m} \gamma_j = 1 \end{cases}$$

$$\iff (\exists \lambda_i, \gamma_j \in \mathbb{R}_{\geq 0}) \begin{cases} \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} = \sum_{i=1}^{t} \lambda_i \begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix} + \sum_{j=1}^{m} \gamma_j \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix} \\ \sum_{j=1}^{m} \gamma_j = 1 \end{cases}$$

$$\iff \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in \text{ConicalHull}\left(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \leq i \leq t, 1 \leq j \leq m\right)$$

□

Decomposition theorem for polyhedron (5/6)

Proof.

1. Denote by S the above cone, that is:

$$S = \text{ConicalHull}\left(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \leq i \leq t, 1 \leq j \leq m\right).$$

□

Decomposition theorem for polyhedron (5/6)

Proof.

1. Denote by S the above cone, that is:

$$S = \text{ConicalHull}\left(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \leq i \leq t, 1 \leq j \leq m\right).$$

2. Apply Farkas-Minkowski-Weyl theorem to S.

□

Decomposition theorem for polyhedron (5/6)

Proof.

1. Denote by S the above cone, that is:
$$S = \text{ConicalHull}(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \leq i \leq t, 1 \leq j \leq m).$$
2. Apply Farkas-Minkowski-Weyl theorem to S.
3. Hence, there exists a matrix A and a vector $\mathbf{b}$ such that
$S = \{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} + \lambda\mathbf{b} \leq 0\}$.

□

Decomposition theorem for polyhedron (5/6)

Proof.

1. Denote by S the above cone, that is:

$$S = \text{ConicalHull}\left(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \le i \le t, 1 \le j \le m\right).$$

2. Apply Farkas-Minkowsi-Weyl theorem to S.
3. Hence, there exists a matrix A and a vector $\mathbf{b}$ such that $S = \{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} + \lambda\mathbf{b} \le 0\}$.
4. Therefore, we have:

$$\begin{aligned} \mathbf{y} \in P &\iff \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in S \\ &\iff \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in \{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} + \lambda\mathbf{b} \le 0\} \\ &\iff A\mathbf{y} \le -\mathbf{b}. \end{aligned}$$

□

Decomposition theorem for polyhedron (5/6)

Proof.

1. Denote by S the above cone, that is:
$$S = \text{ConicalHull}\left(\begin{pmatrix} \mathbf{r}_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_j \\ 1 \end{pmatrix}, 1 \le i \le t, 1 \le j \le m\right).$$
2. Apply Farkas-Minkowsi-Weyl theorem to S.
3. Hence, there exists a matrix A and a vector $\mathbf{b}$ such that $S = \{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} + \lambda\mathbf{b} \le 0\}$.
4. Therefore, we have:
$$\begin{array}{rcl} \mathbf{y} \in P & \Longleftrightarrow & \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in S \\ & \Longleftrightarrow & \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in \{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} \mid A\mathbf{x} + \lambda\mathbf{b} \le 0\} \\ & \Longleftrightarrow & A\mathbf{y} \le -\mathbf{b}. \end{array}$$

This proves that P is a polyhedron and completes the proof. □

Decomposition theorem for polyhedron (6/6)

Corollary 22

Let $P \subseteq \mathbb{R}^d$. Then, P is a polytope if and only if it is a bounded polyhedron.

Proof.

1. Assume P is a polytope.

□

Decomposition theorem for polyhedron (6/6)

Corollary 22

Let $P \subseteq \mathbb{R}^d$. Then, P is a polytope if and only if it is a bounded polyhedron.

Proof.

1. Assume P is a polytope.
   a. Then, it is bounded as the it is the convex hull of finitely many points.

□

Decomposition theorem for polyhedron (6/6)

Corollary 22

Let $P \subseteq \mathbb{R}^d$. Then, P is a polytope if and only if it is a bounded polyhedron.

Proof.

1. Assume P is a polytope.
   a. Then, it is bounded as the it is the convex hull of finitely many points.
   b. Applying the decomposition theorem, and using the trivial cone (reduced to $\{\mathbf{0}\}$, P is also a polyhedron.

□

Decomposition theorem for polyhedron (6/6)

Corollary 22

Let $P \subseteq \mathbb{R}^d$. Then, P is a polytope if and only if it is a bounded polyhedron.

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□

Decomposition theorem for polyhedron (6/6)

Corollary 22

Let $P \subseteq \mathbb{R}^d$. Then, P is a polytope if and only if it is a bounded polyhedron.

Proof.

1. Assume P is a polytope.
   a. Then, it is bounded as the it is the convex hull of finitely many points.
   b. Applying the decomposition theorem, and using the trivial cone (reduced to $\{\mathbf{0}\}$, P is also a polyhedron.
2. Assume P is a bounded polyhedron.
   a. Then, applying the decomposition theorem, and observing that the only bounded cone is $\{\mathbf{0}\}$, we deduce that P is a polytope.

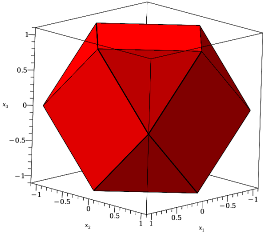
□

Plan

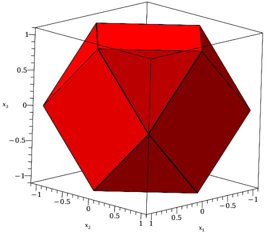
Plan

$$
\text{null}\begin{cases}
-x_3 \le 1 \\
-x_1 - x_2 - x_3 \le 2 \\
-x_1 + x_2 - x_3 \le 2 \\
x_1 - x_2 - x_3 \le 2 \\
x_1 + x_2 - x_3 \le 2 \\
x_3 0 \le 1 \\
-x_1 - x_2 + x_3 \le 2 \\
-x_1 + x_2 + x_3 \le 2 \\
x_1 - x_2 + x_3 \le 2 \\
x_1 + x_2 + x_3 \le 2 \\
-x_2 0 \le 1 \\
x_2 \le 1 \\
-x_1 \le 1 \\
x_1 0 \le 1
\end{cases}
$$

$$\left\{ \text{null} \begin{aligned} -x_3 &\le 1 \\ -x_1 - x_2 - x_3 &\le 2 \\ -x_1 + x_2 - x_3 &\le 2 \\ x_1 - x_2 - x_3 &\le 2 \\ x_1 + x_2 - x_3 &\le 2 \\ x_3 0 &\le 1 \\ -x_1 - x_2 + x_3 &\le 2 \\ -x_1 + x_2 + x_3 &\le 2 \\ x_1 - x_2 + x_3 &\le 2 \\ x_1 + x_2 + x_3 &\le 2 \\ -x_2 0 &\le 1 \\ x_2 &\le 1 \\ -x_1 &\le 1 \\ x_1 0 &\le 1 \end{aligned} \right.$$

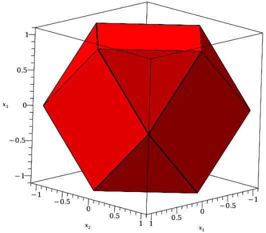


$$\text{null}\begin{cases} -x_3 \leq 1 \\ -x_1 - x_2 - x_3 \leq 2 \\ -x_1 + x_2 - x_3 \leq 2 \\ x_1 - x_2 - x_3 \leq 2 \\ x_1 + x_2 - x_3 \leq 2 \\ x_3 0 \leq 1 \\ -x_1 - x_2 + x_3 \leq 2 \\ -x_1 + x_2 + x_3 \leq 2 \\ x_1 - x_2 + x_3 \leq 2 \\ x_1 + x_2 + x_3 \leq 2 \\ -x_2 0 \leq 1 \\ x_2 \leq 1 \\ -x_1 \leq 1 \\ x_1 0 \leq 1 \end{cases}$$

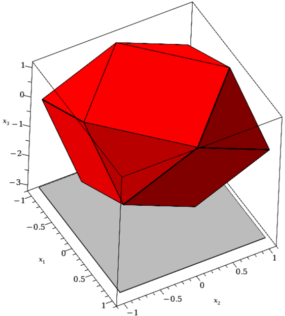


$$\text{null}\begin{cases} 0 \leq 1 + x_2 \\ 0 \leq 1 - x_2 \\ 0 \leq x_1 + 1 \\ 0 \leq 1 - x_1 \end{cases}$$

$$
\text{null}\begin{cases}
-x_3 \leq 1 \\
-x_1 - x_2 - x_3 \leq 2 \\
-x_1 + x_2 - x_3 \leq 2 \\
x_1 - x_2 - x_3 \leq 2 \\
x_1 + x_2 - x_3 \leq 2 \\
x_3 0 \leq 1 \\
-x_1 - x_2 + x_3 \leq 2 \\
-x_1 + x_2 + x_3 \leq 2 \\
x_1 - x_2 + x_3 \leq 2 \\
x_1 + x_2 + x_3 \leq 2 \\
-x_2 0 \leq 1 \\
x_2 \leq 1 \\
-x_1 \leq 1 \\
x_1 0 \leq 1
\end{cases}
$$



$$
\text{null}\begin{cases}
0 \leq 1 + x_2 \\
0 \leq 1 - x_2 \\
0 \leq x_1 + 1 \\
0 \leq 1 - x_1
\end{cases}
$$



$$
\text{null}\begin{cases}
-x_3 \leq 1 \\
-x_1 - x_2 - x_3 \leq 2 \\
-x_1 + x_2 - x_3 \leq 2 \\
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x_3 0 \leq 1 \\
-x_1 - x_2 + x_3 \leq 2 \\
-x_1 + x_2 + x_3 \leq 2 \\
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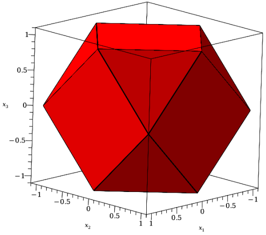
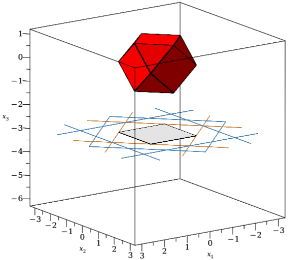
x_1 + x_2 + x_3 \leq 2 \\
-x_2 0 \leq 1 \\
x_2 \leq 1 \\
-x_1 \leq 1 \\
x_1 0 \leq 1
\end{cases}
$$



$$
\text{null}\begin{cases}
0 \leq 1 + x_2 \\
0 \leq 1 - x_2 \\
0 \leq x_1 + 1 \\
0 \leq 1 - x_1
\end{cases}
$$



Application of FME: code generation

```
for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i]*b[j];
}
```

Application of FME: code generation

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for(i=0; i<=n; i++){
   c[i] = 0; c[i+n] = 0;
   for(j=0; j<=n; j++)
      c[i+j] += a[i]*b[j];
}
```

Dependence analysis yields:
$(t, p) := (n - j, i + j)$.

Application of FME: code generation

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for(i=0; i<=n; i++){
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}
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Dependence analysis yields:
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skip slide

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for(i=0; i<=n; i++){
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```
parallel_for (p=0; p<=2*n; p++){
  c[p] = 0;
  for (t=max(0,n-p);
       t<=min(n,2*n-p);t++)
    c[p] += A[t+p-n] * B[n-t];
}
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Polyhedral sets

1. A polyhedral set $P \subseteq \mathbb{Q}^n$ is any $\{\mathbf{x} \in \mathbb{Q}^n \mid A\mathbf{x} \leq \mathbf{b}\}$, where $A \in \mathbb{Q}^{m \times n}$ and $\mathbf{b} \in \mathbb{Q}^m$. Such a linear system is called an H-representation of P.

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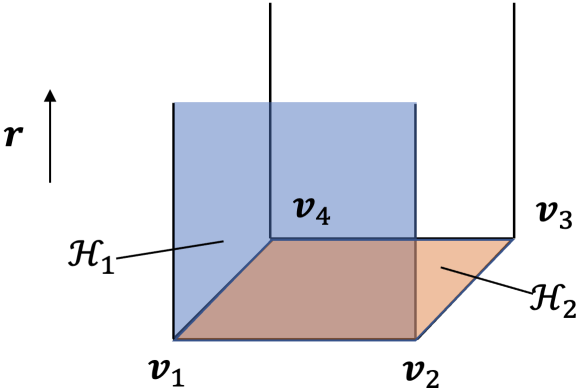
10. Every polyhedral cone has a unique representation as a conical hull of its extremal generators, called the extreme rays of P.
11. Since P is pointed, an extreme ray of P is a one-dimensional face of $\mathrm{CharCone}(P)$.
12. Let V and R denote the set of vertices and extreme rays of P. Then, the pair $\mathrm{Dual}(F) := (V, R)$ is called a V-representation of P.

An unbounded polyhedral set and its representations



The open cube $P := \{(x, y, z) \mid -z \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1\}$ shown above has 4 vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and extreme ray $\mathbf{r}$.

Redundant inequalities, the saturation matrix

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The saturation matrix of F is the $0-1$ matrix $S \in \mathbb{Q}^{m\times k}$, where $S_{i,j} = 1$ iff the j-th element of $\mathrm{Dual}(F)$ saturates the i-th inequality of F.

A bounded polyhedral set and its the saturation matrix

F

$\ell_1 : x + y \leq 1$

null$\ell_2 : -x - y \leq 1$

$\ell_3 : x - y \leq 1$

$\ell_4 : -x + y \leq 1$

$\mathrm{Dual}(F)$

$\mathbf{v}_1 : (0, 1)$

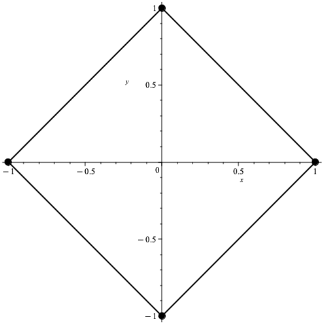
null$\mathbf{v}_2 : (1, 0)$

$\mathbf{v}_3 : (-1, 0)$

$\mathbf{v}_4 : (0, -1)$

satM(F)

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
ℓ_1	1	1	0	0
ℓ_2	0	0	1	1
ℓ_3	0	1	0	1
ℓ_4	1	0	1	0



Basic redundancy check

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3. The affine rank of $\mathrm{Dual}(F) \cap \mathcal{H}_\ell$ is the rank of the matrix
$$[\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \ldots, \mathbf{v}_t - \mathbf{v}_1, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s].$$

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With these notations, we have the following lemma. We note that any permutation $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t)$ would leave this result unchanged.

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1. Denote by $\mathrm{Dual}(F) \cap \mathcal{H}_\ell$ the vertices and rays in $\mathrm{Dual}(F)$ saturating the hyperplane $\mathcal{H}_\ell$.
2. Write $\mathrm{Dual}(F) \cap \mathcal{H}_\ell := (\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}, \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s\})$, where $\mathbf{v}_i$'s are vertices and $\mathbf{r}_j$'s are rays.
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- Consider the elimination of a variable, say x, during FME.

skip slide

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- Then, we have

$$\mathrm{proj}(\{\ell_{pos}, \ell_{neg}\}, \{x\}) = \{-a_2 \mathbf{c}_1^t \mathbf{y} + a_1 \mathbf{c}_2^t \mathbf{y} \leq -a_2 b_1 + a_1 b_2\}.$$

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After computing all $\mathrm{proj}(\{\ell_{pos}, \ell_{neg}\}, \{x\})$'s and eliminating the redundant such inequalities, how to update the saturation matrix and prepare for the next variable elimination?

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We have:

$$\mathcal{S}^{\mathcal{VR}}(\mathrm{proj}(\{\ell_{pos}, \ell_{neg}\}, \{x\})) = \mathrm{proj}(\mathcal{S}^{\mathcal{VR}}(\ell_{pos}) \cap \mathcal{S}^{\mathcal{VR}}(\ell_{neg}), \{x\}).$$

skip slide

Algorithm 1: CheckRedundancy

Input: 1. the inequality system F with m inequalities;
2. the saturation matrix satM.

Output: the minimal system F_{irred} and the corresponding saturation matrix $\mathrm{satM}_{\mathrm{irred}}$.

Irredundant := $\{\mathrm{seq}(i, i = 1..m)\}$.
for i from 1 to m **do**
    if the number of nonzero elements in $\mathrm{satM}[i]$ is less than n **then**
        Irredundant := *Irredundant* $\setminus \{i\}$.
        next.
    for j in *Irredundant* $\setminus \{i\}$ **do**
        if $\mathrm{satM}[i] = \mathrm{satM}[i] \& \mathrm{satM}[j]$ **then**
            Irredundant := *Irredundant* $\setminus \{i\}$.
            break.
F_{irred} := $[\mathrm{seq}(F[i], i \text{ in } \textit{Irredundant})]$ and
$\mathrm{satM}_{\mathrm{irred}}$:= $[\mathrm{seq}(\mathrm{satM}[i], i \text{ in } \textit{Irredundant})]$.
return F_{irred} and $\mathrm{satM}_{\mathrm{irred}}$.

Algorithm 2: Minimal projected representation

Input: 1. an inequality system F;
2. a variable order $x_1 > x_2 > \ldots > x_n$.

Output: the minimal projected representation *res* of F.

Compute the V-representation V of F by DD method;
Set $res :=$ table().
Sort the elements in V w.r.t. the reverse lexico order.
Compute the saturation matrix satM.
F, satM := $\mathrm{CheckRedundancy}(F, \mathrm{satM}(F))$.
$res[x_1] := F^{x_1}$.
for i from 1 to $n-1$ **do**
    $(F^p, F^n, F^0) :=$ partition(F).
    $V_{new} :=$ proj$(V, \{x_i\})$.
    Merging: satM := Merge(satM).
    Let $F_{new} := F^0$ and $\mathrm{satM}_{new} := \mathrm{satM}[F^0]$.
    foreach $f_p \in F^p$ and $f_n \in F^n$ **do**
        Append proj$((f_p, f_n), \{x_i\})$ to F_{new},
        Append $\mathrm{satM}[f_p]\&\mathrm{satM}[f_n]$ to satM_{new}.
    F, satM := $\mathrm{CheckRedundancy}(F_{new}, \mathrm{satM}_{new})$.
    $V := V_{new}$, $res[x_{i+1}] := F^{x_{i+1}}$.
return *res*.

Plan

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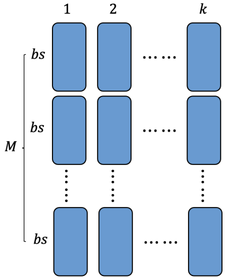

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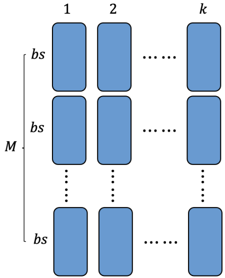

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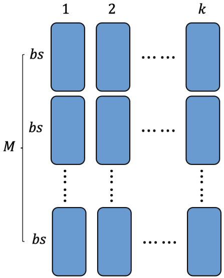

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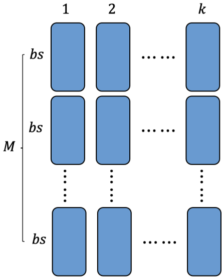
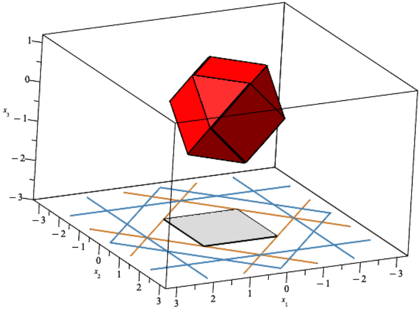
4. For cache complexity reasons, we maintain both satM(F) and satM$(F)^t$.
5. Moreover, these matrices should be represented by blocks.
6. Other key tasks Algorithm 2 are
   - computing the V-representation of each successive projection
   - updating the saturation matrix.



Plan

Cuboctahedron



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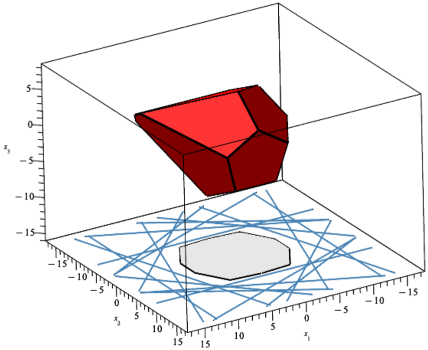
2. **weakly redundant inequalities eliminated by cardinality**
3. **weakly redundancies inequalities eliminated by containment**

Snub disphenoid (triangular dodecahedron)



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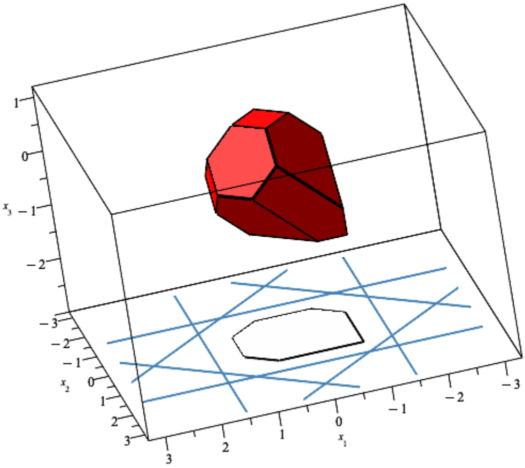
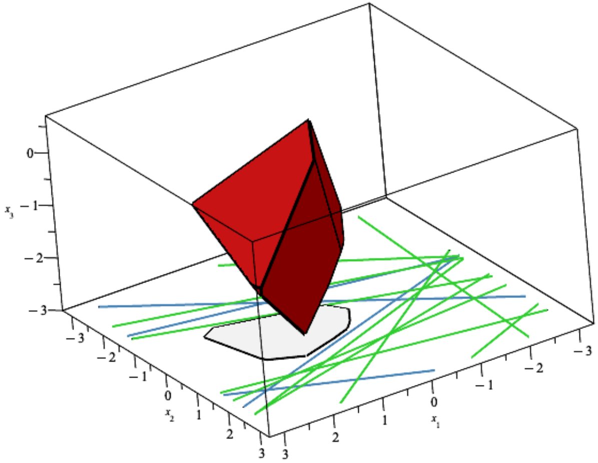
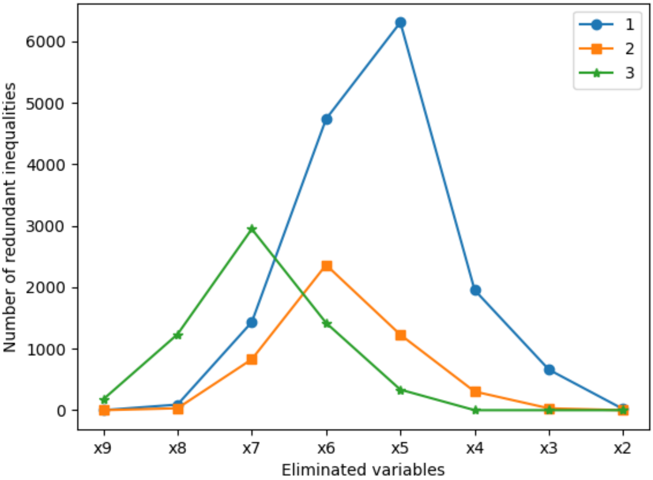
Truncated octahedron



Random 3D polyhedron



Random 10D polyhedron



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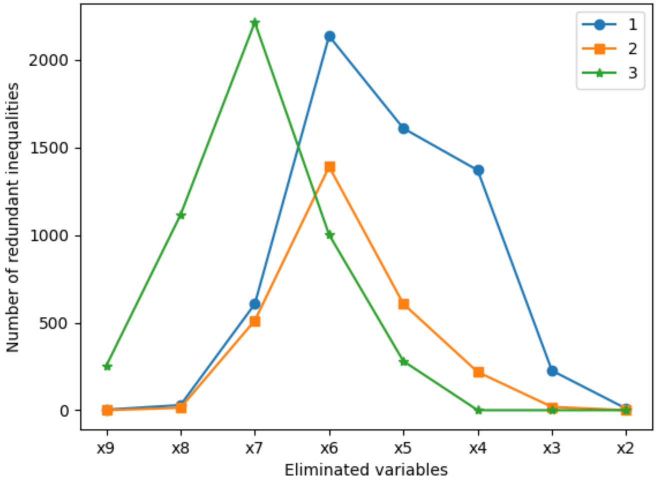

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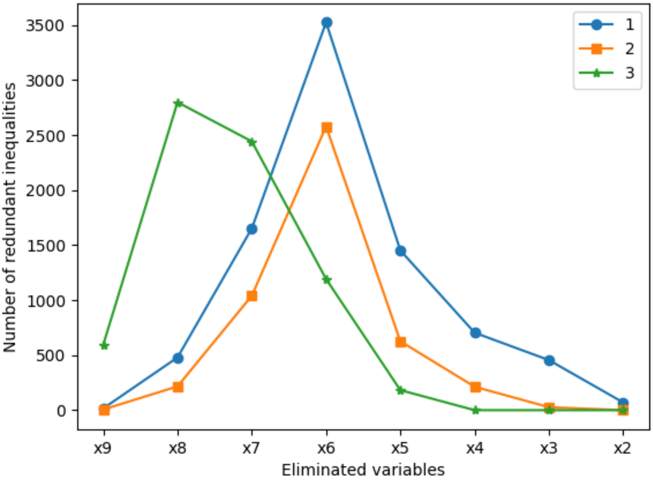

Comparative experimentation (1/3)

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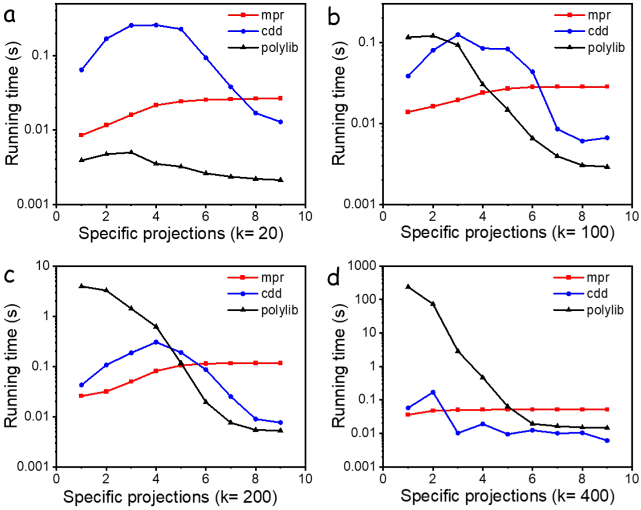
All the experimental results were collected on a PC (Intel(R) Xeon(R) Gold 6258R CPU 2.70GHz, 503G RAM, Ubuntu 20.04.3).

Comparative experimentation (2/3)



1. Four different random polyhedra with $m = 15$ and $n = 10$.
2. For $1 \leq i \leq 9$, in the hor. axiss, the first i variables are eliminated.
3. The vert. axis in each figure shows the running time (in seconds).

test case	(n, m, k)	mpr (msec.)	BPAS (msec.)	cdd (msec.)	polylib (msec.)
32hedron	(6, 32, 11)	6.54	16.80	4183.08	**1.92**
64hedron	(7,64,13)	13.05	52.42	>5min	**1.67**
francois	(13,27,2304)	499.92	**253.66**	388.36	> 5min
francois2	(13,31,384)	**41.80**	140.34	55.17	80.63
herve.in	(14,25,262)	34.42	140.34	294.01	**30.08**
c6.in	(11,17,31)	**9.85**	12.72	84.11	5.56
c9.in	(16,18,140)	**25.08**	65.54	151.17	131.53
c10.in	(18,20,142)	22.10	98.68	249.02	**16.06**
S24	(24, 25,25)	23.50	58.80	748.67	**17.47**
S35	(35, 36,36)	46.55	182.14	3575.00	**46.007**
cube	(10, 20,1024)	**81.33**	201.92	125.900	161.06
C56	(5, 6,6)	3.67	4.09	11.81	**0.79**
C1011	(10, 11,11)	24.99	115.68	1716.25	**9.99**
C510	(5, 42,10)	12.00	40.01	>5min	**4.42**
T1	(5, 10,38)	**5.61**	16.44	27.42	8.81
T3	(10,12,29)	21.29	141.64	288.07	**12.07**
T5	(5, 10,36)	8.12	15.62	22.92	**4.76**
T6	(10,20,390)	**1142.9**	23800.11	14937.61	>5min
T7	(5, 8,26)	5.81	10.79	13.96	**4.00**
T9	(10,12,36)	**36.56**	414.53	479.18	100.34
T10	(6, 8,24)	**4.58**	13.65	18.39	5.27
T12	(5, 11,42)	**8.52**	19.03	38.65	8.60
R_15_20	(15, 20,1328)	**28430.40**	336035.00	38037.21	>5min

Complexity estimates (1/2)

Recall the notations

1. m is the number of inequalities and n is the dimension of the ambient space. If the input H-representation is irredundant, the m is also the number of facets of P.
2. Let $h := \text{height}([A, \mathbf{b}])$, let θ be the coefficient of linear algebra and ω the bit-size of a machine word.

Well-known bounds

1. The size k of the V-representation (V, R) is at most $\binom{m}{n} + \binom{m}{n-1} \leq \frac{m^n}{n!}$.
2. From [8, 9] for $1 \leq i < n$, after eliminating i variables during the process of FME, the number of irredundant inequalities defining the projection is at most $\binom{m}{n-i-1} \leq m^n$.

Theorem 28

The costs for computing all the inequalities (redundant and irredundant) and generating the initial saturation matrix are within $O(m^{2n} n^{\theta+\varepsilon} h^{1+\varepsilon})$ bit operations, while the costs for updating and operating on the saturation matrices are bounded over by $\frac{3m^{3n-4}}{\omega}$ word operations.

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Bounds for FME

1. FME based on LP: $O(n^2\, m^{2n}\, \text{LP}(n, 2^n h n^2 m^n))$ bit operations, where $\text{LP}(d, H)$ is an upper bound for the number of bit operations required for solving a linear program in d variables and with total bit size H. For instance, in the case of Karmarkar's algorithm [12], we have $\text{LP}(d, H) \in O(d^{3.5} H^2 \cdot \log H \cdot \log\log H)$.
2. FME based on redundancy test cone: $O(m^{\frac{5n}{2}}\, n^{\theta+1+\epsilon} h^{1+\epsilon})$ bit operations, for any $\epsilon > 0$.
3. This paper: $O(m^{2n} n^{\theta+\varepsilon} h^{1+\varepsilon})$ bit operations and $\frac{3m^{3n-4}}{\omega}$ word operations.

Concluding remarks

Summary and notes

1. We proposed a technique for removing redundant inequalities in linear systems.
2. It relies on the analysis of 3 different types of redundancies
3. Our redundancy tests allow for efficient implementation based on bit-vector arithmetic.
4. From the experimental results, our method works best on hard problems.
5. This is promising to solve large scale problems in areas like information theory, SMT and optimizing compilers.

Work in progress

1. Our implementation has room for improvements.
2. Indeed, our algorithms have opportunities for both multithreaded parallelism and instruction-level parallelism.
3. The third criterion (redundancy test based on containment) needs further study to discover the container.

Plan

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Dependence analysis

Cholesky's LU decomposition:

```
1:   for(i = 1; i <= n; i++){
        x = a[i][i];
        for(k = 1; k < i; k++)
2:          x = x - a[i][k] * a[i][k];
3:      p[i] = 1.0/sqrt(x);
        for(j = i + 1; j <= n; j++){
4:          x = a[i][j];
            for(k = 1; k < i; k++)
5:              x = x - a[j][k] * a[i][k];
6:          a[j][i] = x * p[i];
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system 1:

$$\left\{\begin{array}{lr} & 1 \le i \le n \\ & i+1 \le j \le n \\ & 1 \le k \le i-1 \\ \text{null} & 1 \le i' \le n \\ & i'+1 \le j' \le n \\ & j = j', k = i' \\ & i < i' \end{array}\right.$$

system 2:

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system 3:

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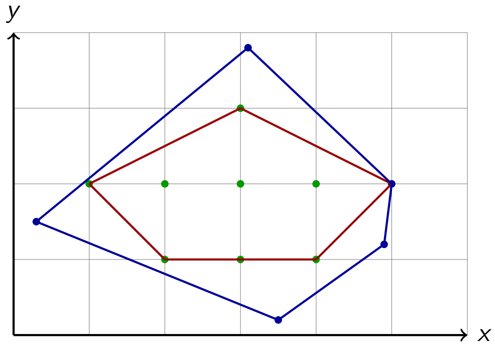
Plan

Integer Hull

1. The **integer hull** of the polyhedron $P \subseteq \mathbb{Q}^d$, denoted by P_I, is the smallest convex polyhedron containing all the integer points of P.



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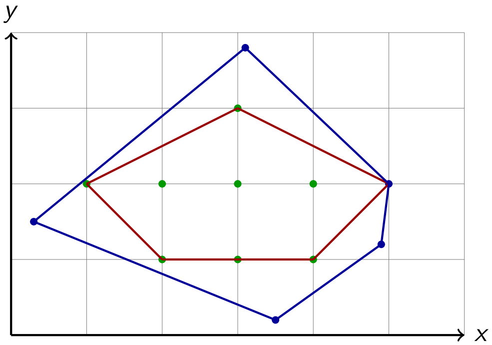

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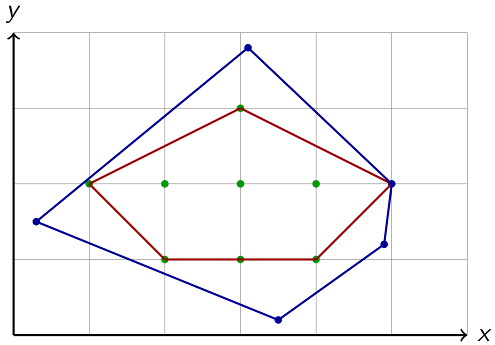

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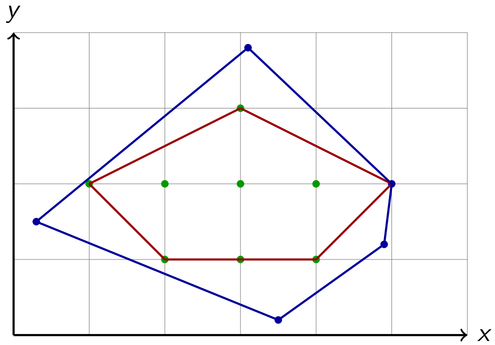
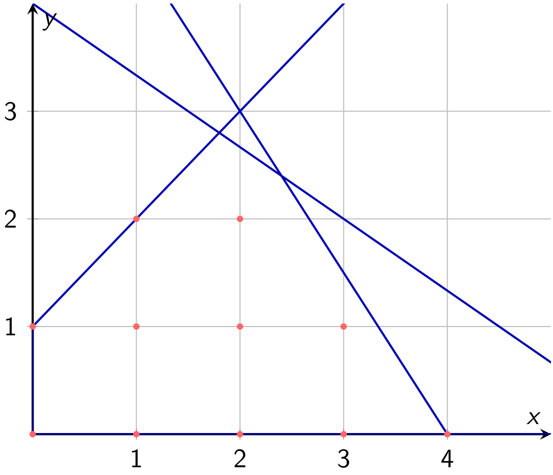
4. Thus, for P pointed, the convex hull of all the vertices of P_I is P_I itself.



Integer hull: simple example

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ 3\,x + 2\,y & \leq & 12 \\ 2\,x + 3\,y & \leq & 12 \\ -x + y & \leq & 1 \end{cases}$$

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ y & \leq & 2 \\ x + y & \leq & 4 \\ -x + y & \leq & 1 \end{cases}$$



Figure

Lattices

1. A subset $L \subseteq \mathbb{Z}^d$ is called an **integer lattice** (or simply a lattice) if
$$L = \{\mathbf{x} \in \mathbb{Z}^d \mid (\exists \mathbf{t} \in \mathbb{Z}^c)\ \mathbf{x} = A\mathbf{t} + \mathbf{b}\}$$
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2. It is convenient to see this lattice as the solution set of the systems of congruence relations
$$\mathbf{x} \equiv \mathbf{b} \mod A.$$

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   - b. $\{\mathbf{x} \in \mathbb{Z}^d \mid C\mathbf{x} = \mathbf{q}\} = \{U_R H^{-1}\mathbf{q} + U_L\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^{d-r}\}$.
6. These results generalize to the case where the rank of C is arbitrary, see [21],

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4. The algorithm IntegerPointDecomposition [11] decomposes any $\mathbb{Z}$-polyhedron into **normalized** $\mathbb{Z}$-polyhedra.

Plan

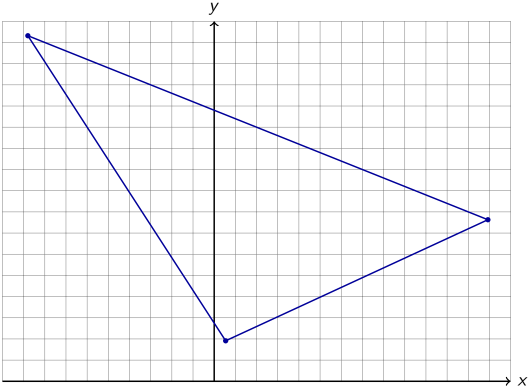
Example (0/3)

Input

Let us look at a simple example first.

Vertices: $(-44/5, 408/25), (349/27, 206/27), (85/57, 109/57)$

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -7 \end{cases}$$



Example (1/3)

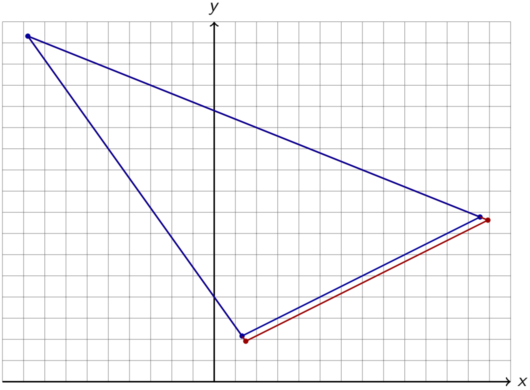
Normalization

Replace the facets that could not have integer point

Vertices: $(-44/5, 408/25)$, ~~$(349/27, 206/27)$, $(85/57, 109/57)$,~~ $(113/9, 70/9)$, $(25/19, 41/19)$

~~$3x - 6y \leq -7$~~

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -9 \end{cases}$$

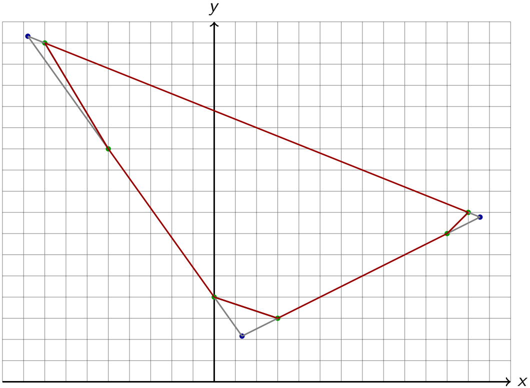


Example (2/3)

Partition

Vertices: $(-44/5, 408/25), (113/9, 70/9), (25/19, 41/19)$
Find the triangles with vertices: $[(-8, 16), (-44/5, 408/25), (-5, 11)]$, $[(3, 3), (25/19, 41/19), (0, 4)]$, $[(12, 8), (113/9, 70/9), (11, 7)]$

$$\begin{cases} 5y & \leq & -2x + 64 \\ 5y & \geq & -7x + 20 \\ 2y & \geq & x + 3 \end{cases}$$



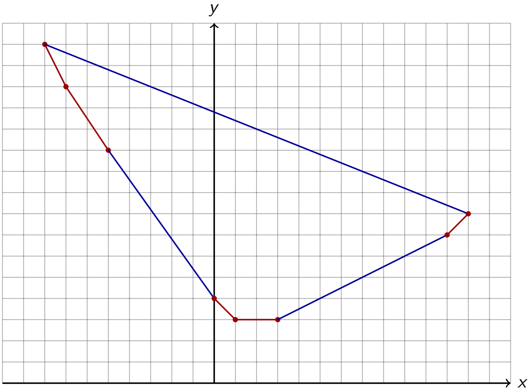
Example (3/3)

Merging

Vertices: $(-8, 16), (-7, 14), (-5, 11), (0, 4), (1, 3), (3, 3), (11, 7), (12, 8)$

$$\begin{cases} 5y & \leq & -2x + 64 \\ 5y & \geq & -7x + 20 \\ 2y & \geq & x + 3 \end{cases}$$

$$\begin{cases} y & \geq & -2x \\ 2y & \geq & 3x + 7 \\ y & \geq & -x + 4 \\ y & \geq & 3 \\ y & \geq & x - 4 \end{cases}$$



Main steps of our algorithm

Our algorithm has 3 main steps:

- **Normalization**: construct a new polyhedral set Q from P as follows. Consider in turn each facet F of P:
  1. if the hyperplane H supporting F contains an integer point, then H is a hyperplane supporting a facet of Q,
  2. otherwise we slide H towards the center of P along the normal vector of F, stopping as soon as we hit a hyperplane H' containing an integer point, then making H' a hyperplane supporting a facet of Q.

  Clearly $Q_I = P_I$.

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- **Partitioning**: make each part of the partition a polyhedron R which:
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- **Merging**: Once the integer hull of each part of the partition is computed and given by the list of its vertices, an algorithm for computing the convex hull of a set points, such as `QuickHull`, can be applied to deduce P_I.

The general algorithm on a 3D example

Normalization

The integer hull of the normalized polyhedral set should be the same as that of the input

$$\left\{\begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -5405x_1 + 4965x_2 + 3870x_3 & \leq & 4303504 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$

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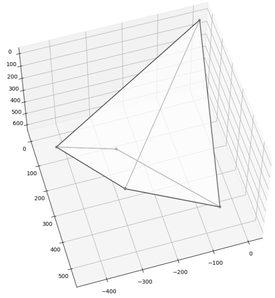

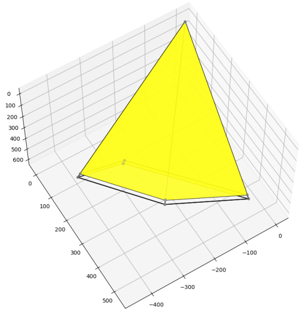

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The general algorithm: building the partition

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For each face f (of positive dimension) of P:

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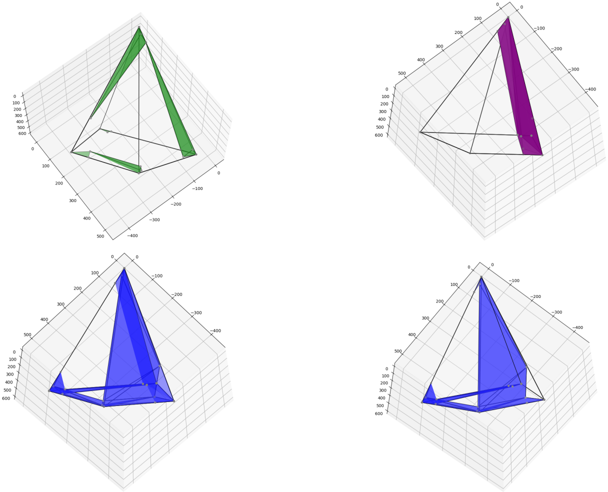
4. See [15] and the PhD thesis of Lin-Xiao Wang.

The general algorithm on a 3D example

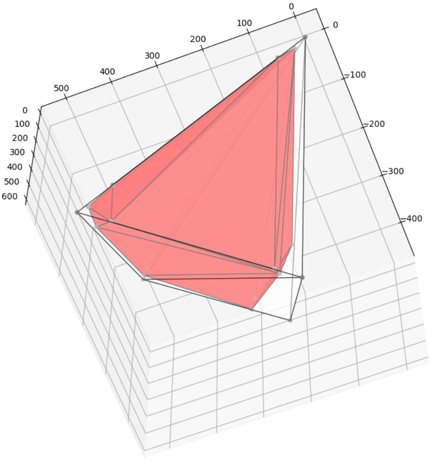
Partition

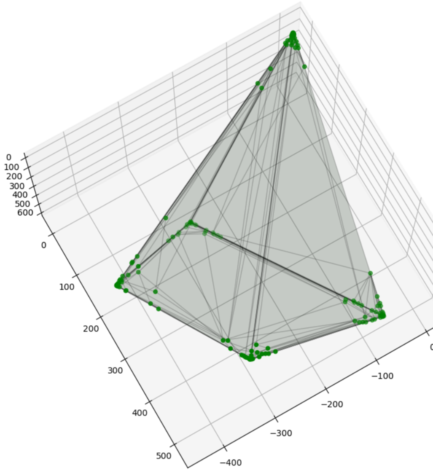


The general algorithm on a 3D example

Merging

The integer hull has 139 vertices





“Closest integer points” on a facet to each of its vertices

Projection and recursive call

In $\mathbb{Q}^d$, for a facet F of dimension $d-1$, and its vertex set V:

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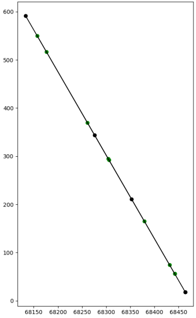
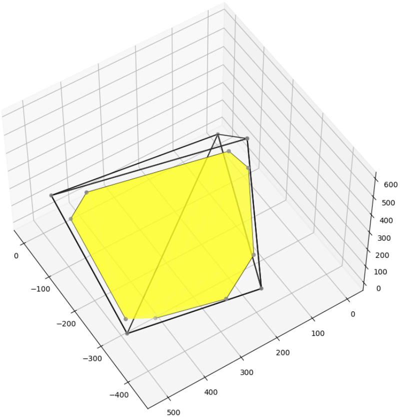
6. finally, we find in V_I the “closest integer points” to each v of V.

Closest integer points on a face to one of its vertices

Projection and recursive call

$$R_F : \begin{cases} x_1 = 993x_1' + 573x_2' - 67995300 \\ x_2 = 1081x_1' + 623x_2' - 74020200 \\ x_3 = x_2' \end{cases}$$





The `PolyhedralSets:-IntegerHull` command in MAPLE

```
> with(PolyhedralSets) :
> ineqs := [2 x + 5 y ≤ 64, 7 x + 5 y ≥ 20 , 3 x − 6 y ≤ -7] :
> poly := PolyhedralSet(ineqs, [x, y]);
```

$$poly := \begin{cases} Coordinates & : & [x, y] \\ Relations & : & \left[-x - \frac{5\,y}{7} \leq -\frac{20}{7},\, x - 2\,y \leq -\frac{7}{3},\, x + \frac{5\,y}{2} \leq 32 \right] \end{cases}$$

```
> IntegerHull(poly);
```

$$[[[12, 8], [-8, 16], [-7, 14], [-5, 11], [0, 4], [1, 3], [3, 3], [11, 7]], [\ ]]$$

```
> IntegerHull(poly, returntype=polyhedralset);
```

$$\begin{cases} Coordinates & : & [x, y] \\ Relations & : & \left[-y \leq -3,\ -x - y \leq -4,\ -x - \frac{5\,y}{7} \leq -\frac{20}{7},\ -x - \frac{2\,y}{3} \leq -\frac{7}{3},\ -x - \frac{y}{2} \leq 0,\, x - 2\,y \leq - \right. \end{cases}$$

```
>
```

The PolyhedralSets:-IntegerHull command in Maple

```
> restart; with(PolyhedralSets) :
> vertices := [[10, 10, 10, 10/3], [-140/8, -220/12, -10, -10/3], [60/8, 20, -100/12, -70/3], [
      /4, -100/12, 70/2, 35/3], [0, 0, 0, 50/3]] :
  vars := [x1, x2, x3, x4] :
  poly := PolyhedralSet(vertices, [ ], vars);
```

$$poly := \begin{cases} Coordinates : [x1, x2, x3, x4] \\ Relations : \left[-x1 + \frac{503\, x2}{694} + \frac{85\, x3}{694} + \frac{311\, x4}{2082} \leq \frac{7775}{3123}, -x1 + \frac{2715\, x2}{2234} + \frac{603\, x3}{2234} \right. \end{cases}$$

```
> IntegerHull(poly);
```

[[[−15, −16, −6, −2], [−15, −15, −9, −4], [−14, −15, −4, −1], [−13, −13, −8, −1],
[−13, −12, −9, −5], [−12, −13, −4, 1], [−12, −13, −4, 2], [−12, −12, −3, −3], [−1
−12, −3, −1], [−11, −11, −6, −3], [−11, −11, −1, −3], [−10, −8, −8, −5], [−9, −
−8, −8], [−7, −7, −4, 7], [−7, −6, −5, 3], [−7, −3, −8, −10], [−7, −3, −7, −10],
[−6, −4, −5, 0], [−5, −9, 23, 8], [−5, −4, −4, 5], [−5, −4, −3, −2], [−4, −5, 3, 10],
[−4, −5, 3, 11], [−4, −3, −2, −3], [−4, 0, −6, −9], [−3, −8, 30, 10], [−3, −7, 23, 10
[−3, −7, 23, 11], [−3, −6, 24, 6], [−3, 3, −8, −12], [−3, 3, −7, −13], [−2, −7, 31, 11
[−2, −6, 24, 8], [−2, −6, 24, 12], [−2, −6, 26, 11], [−2, −5, 25, 7], [−2, −5, 26, 6],

The PolyhedralSets:-IntegerHull command in MAPLE

```
> ineqs := [-x1-(132*x2)/205-(62*x3)/205 <= -1358/123, -x1 + (34*x2)/34 + (4*x3)/4
      <= 1405/17, x1-(12*x2)/118 + (83*x3)/177 <= 3500/59]:
  poly := PolyhedralSet(ineqs, [x1, x2, x3]);
  IsBounded(poly);
```

$$poly := \begin{cases} Coordinates : [x1, x2, x3] \\ Relations : \left[-x1 - \frac{132\,x2}{205} - \frac{62\,x3}{205} \leq -\frac{1358}{123}, -x1 + x2 + x3 \leq \frac{1405}{17}, x1 - \frac{6\,x2}{59}\right. \end{cases}$$

false

```
> IntegerHull(poly);
```

$$\Big[[[-20, 36, 26], [-4, -25, 103], [-2, -30, 107], [-1, -36, 117], [0, -38, 118], [0, -36, 118], [1, -37, 112], [1, -34, 117], [2, -39, 113], [2, -38, 114], [10, -43, 95], [26, -51, 60], [399, -238, -776], [403, -240, -785], [453, -265, -897], [1544, -811, -3342]], \left[\left[\frac{101}{260}, 1, -\frac{159}{260}\right], \left[\frac{2012}{4509}, -\frac{6041}{27054}, -1\right], \left[-\frac{70}{337}, \frac{267}{337}, -1\right]\right]\Big]$$

Benchmarks 2D

E&C represents "enumeration and convex hull", which in Maple is done by `ZPolyhedralSets:-EnumerateIntegerPoints` and `ConvexHull`. `Normaliz` is an open source tool for computations in affine monoids, vector configurations, lattice polytopes, and rational cones.

Volume	27.95	null	111.79	null	11179.32	null
Algorithm	IntegerHull	E&C	IntegerHull	E&C	IntegerHull	E&C
Maple (ms)	172	410	244	890	159	58083
C/C++ (ms)	0.284	0.768	0.339	1.676	0.286	6.883
Normaliz (ms)	835.730	null	462.116	null	1559.401	null

Table: Integer hulls of triangles

Volume	58.21	null	5820.95	null	23283.82	null
Algorithm	IntegerHull	E&C	IntegerHull	E&C	IntegerHull	E&C
Maple (ms)	303	752	275	31357	304	123159
C/C++ (ms)	0.451	0.565	0.478	0.657	0.396	0.682
Normaliz (ms)	2.837	null	1216.238	null	740.559	null

Table: Integer hulls of hexagons

Benchmarks 3D

Volume	447.48	null	6991.89	null	55935.2	null
Algorithm	IntegerHull	E&C	IntegerHull	E&C	IntegerHull	E&C
Maple (ms)	977	7289	1223	74804	1378	531904
C/C++ (ms)	4.488	0.826	4.615	0.923	4.624	1.527
Normaliz (ms)	851.495	null	956.666	null	793.192	null

Table: Integer hulls of tetrahedrons (4 vertices, 4 facets and 6 edges)

Volume	412.58	null	7050.81	null	60417.63	null
Algorithm	IntegerHull	E&C	IntegerHull	E&C	IntegerHull	E&C
Maple (ms)	1476	5711	1573	60233	1728	512101
C/C++ (ms)	11.049	21.235	16.001	145.068	23.822	2082.559
Normaliz (ms)	7862.109	null	N/A	null	N/A	null

Table: Integer hulls of triangular bipyramids (5 vertices, 6 facets and 9 edges)

Conclusions

1. We implemented the proposed algorithm for in both MAPLE and C/C++.
2. Our algorithm takes into consideration the geometric properties of the input polyhedral set.
3. That is, if the input polyhedral set is close to be its own integer hull, then computations are cheaper
4. Moreover, the cost of the computations depend mainly on the shape of the input while the size of the input has little impact.
5. Doing a complexity analysis that would reflect that fact is an open problem to us.

Plan

Plan

Counting memory accesses in a for-loop nest (1/2)

Consider the well-known example SOR (Successive-Over Relaxation) from the numerical solving of PDEs (Partial differential Equations).

```
for (i=2, i<N, i++) null
    for (j=2, j <N, j++) null
        a[i][j] = (2*a[i][j] + a[i-1][j] + a[i+1][j] + null
                   a[i][j-1] + a[i][j+1])/6; null
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- The memory slots accessed by the for-loop nest are given by:

$$\{(i + \Delta i, j + \Delta j) \quad | \quad -1 \le \Delta i - \Delta j, \Delta i + \Delta j, \le 1, 2 \le i, j \le N - 1\}$$

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- Using standard techniques from Linear Algebra, namely Fourier-Motzkin elimination (FME), we can rewrite the above set as:

  $$\left\{ (x, y) \quad | \quad \left\{ \begin{array}{l} 1 \leq x, y \leq N \\ 3 \leq x + y \leq 2N - 1 \\ 2 - N \leq x - y \leq N - 2 \end{array} \right. \right\}, \text{ for } N \geq 3.$$

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- Hence the problem becomes counting the number of integer points of a parametric polyhedral set P_N.

Counting memory accesses in a for-loop nest (2/2)

```
for (i=2, i<N, i++) null
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```

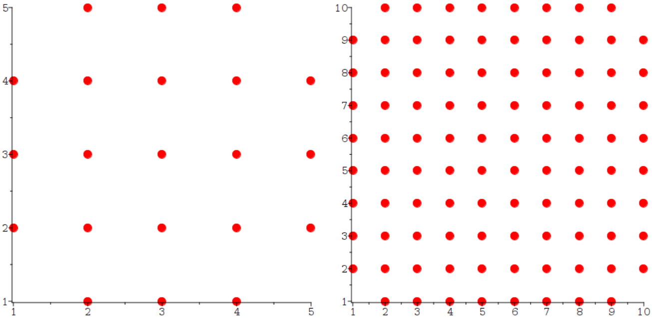

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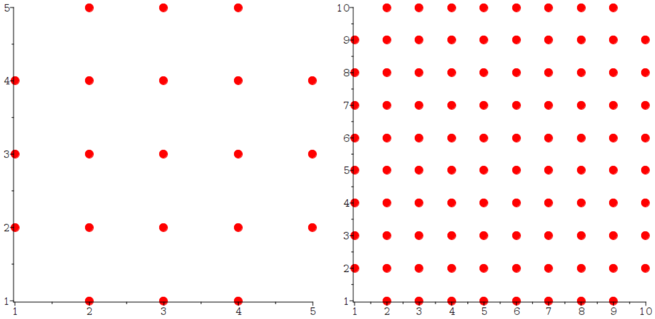

The integer points of the parametric polyhedron P_N for $N = 5$ and $N = 10$.
We will see later that $\mid P \cap \mathbb{Z}^2 \mid = N^2 - 4$.

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1. Given a parametric polyhedron $P(\vec{b})$, we want to count the number of its integer points as a function $c(\vec{b})$ of the parameter $\vec{b}$.

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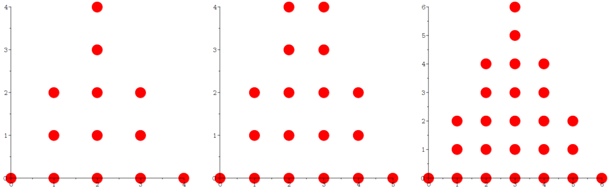

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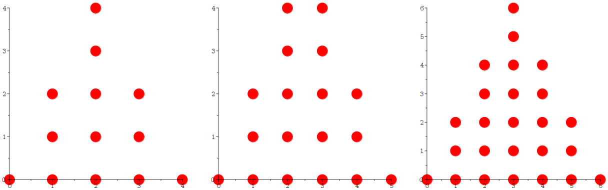

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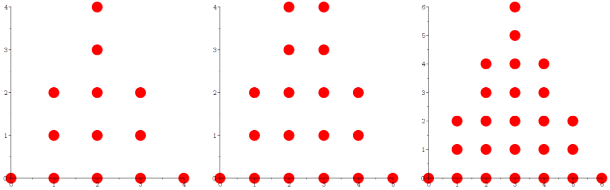

5. Fortunately, Ehrhart Theory tells us that these variations are periodic
6. Hence, the function $c(\vec{b})$ is computable as a piece-wise function.

Related works

- Given a 2D polytope P, whose vertices are integer points, Pick's theorem relates the area A of P, the number b of integer points on the border of P, and the number i in the interior of P:

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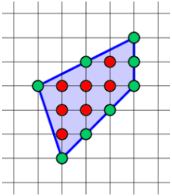

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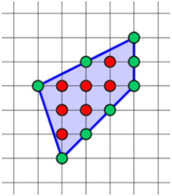

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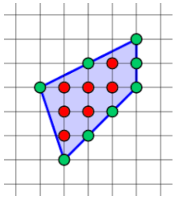

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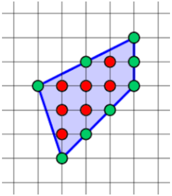

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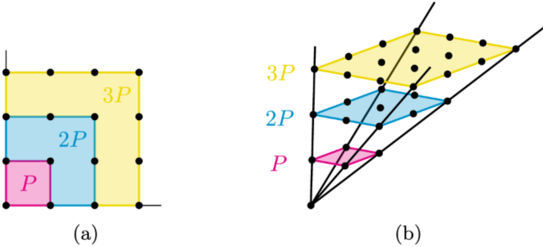

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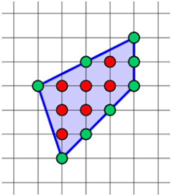

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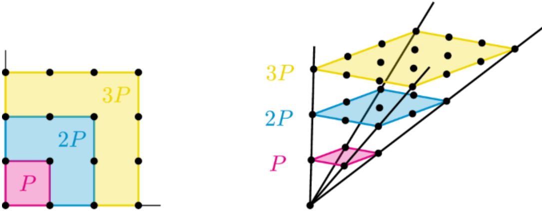
- See Ehrhart polynomial.
- Images are from Wikipedia (fair use category).



(a) (b)

Plan

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- For $d = 2$, suppose P is the ray corresponding to $y = 0$ and $x \geq 0$, then:

$$G(P, \mathbf{x}) =$$

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- When $d = 2$, we write (x, y) instead of (x_1, x_2).

Definition

The **generating function** of P is the formal Laurent series:

$$G(P, \mathbf{x}) = \sum_{\mathbf{e} \in P \cap \mathbb{Z}^d} \mathbf{x}^\mathbf{e}.$$

- If P is bounded, then $G(P, (1, \ldots, 1))$ counts the number of its integer points.
- If P is not bounded, then $G(P, \mathbf{x})$ is a formal power series and can still be manipulated algorithmically.
- For $d = 2$, suppose P is the ray corresponding to $y = 0$ and $x \geq 0$, then:

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Generating function of a polyhedral set (1/4)

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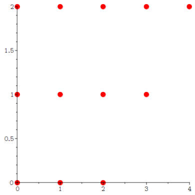
Generating function of a polyhedral set (2/4)

With $d = 2$, we will compute $G(P, \mathbf{x})$ for the polyhedron P given as the convex hull of the 12 points on the figure below.



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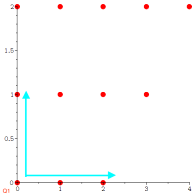

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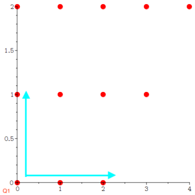

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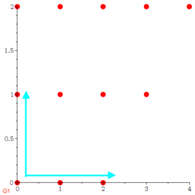

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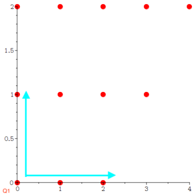

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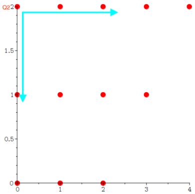

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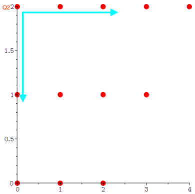

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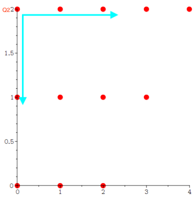

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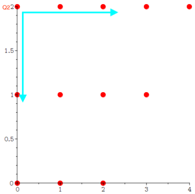

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Generating function of a polyhedral set (2/4)

Continuing with the other corners Q_3 and Q_4 of the polytope P

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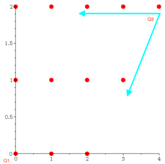

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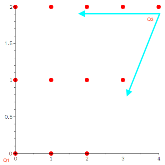

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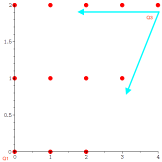

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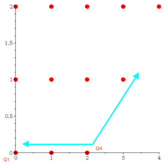

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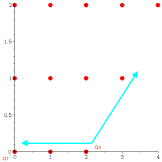

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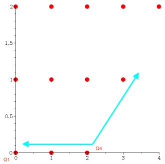

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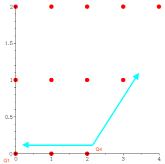

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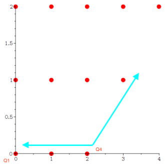

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Generating function of a polyhedral set (2/4)

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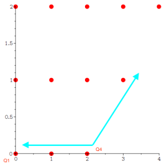

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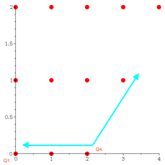
Generating function of a polyhedral set (2/4)

Continuing with the other corners Q_3 and Q_4 of the polytope P



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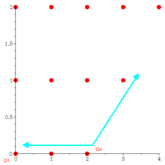
Generating function of a polyhedral set (2/4)

Continuing with the other corners Q_3 and Q_4 of the polytope P



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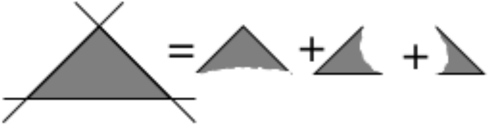
Consequently

$$\begin{array}{rcl} \mid P \cap \mathbb{Z}^2 \mid & = & G(P, (1, 1))) \\ & & = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ & & = 12. \end{array}$$

Recall Brion's formula

- This formula asserts that for a polytope $P \subseteq \mathbb{Q}^d$ its generating function is the sum of the generating functions of its corners (= vertex cones)



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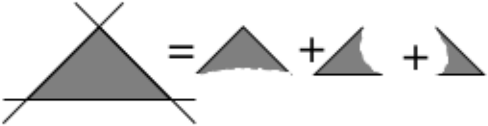

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- Our previous calculations used two facts

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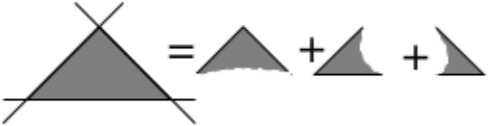

$$G(P, \mathbf{x}) \;=\; G(Q_1, \mathbf{x}) + G(Q_2, \mathbf{x}) + G(Q_3, \mathbf{x})$$

- Our previous calculations used two facts
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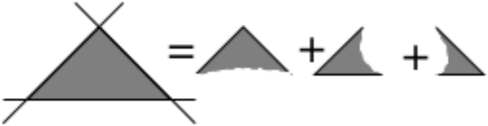
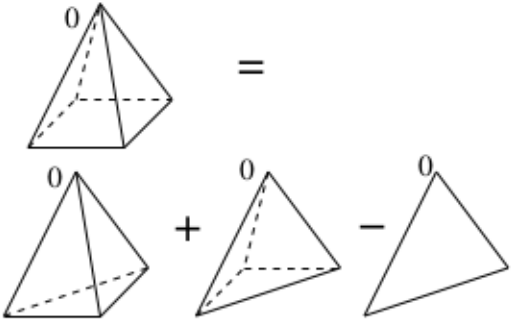


$$G(P, \mathbf{x}) = G(Q_1, \mathbf{x}) + G(Q_2, \mathbf{x}) + G(Q_3, \mathbf{x})$$

- Our previous calculations used two facts
  1. In dimension $d = 2$, every cone is **simplicial** that is, can be generated by d rays,
  2. The cones $Q2$, Q_3, Q_4 are **unimodular**, that is, the sums of the power series $G(Q_2, \mathbf{x})$, $G(Q_3, \mathbf{x})$, $G(Q_4, \mathbf{x})$ can be deduced from that of $G(Q_1, \mathbf{x})$ (the first quadrant) by means of unimodular transformations (that is, mapping integer vectors to integer vectors).

Barvinok's algorithm

- In dimension d, one can decompose any cone into **simplicial** cones (= cones generated by d rays),



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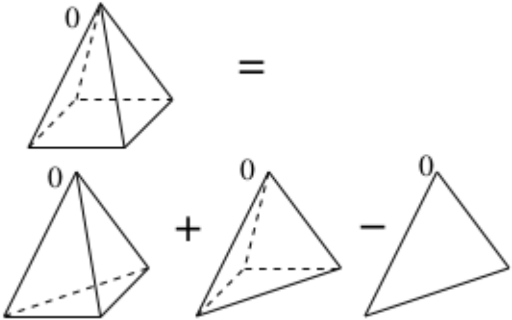

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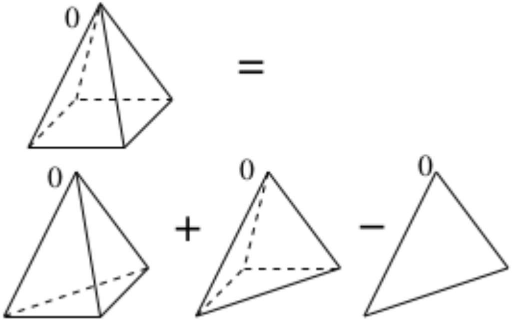

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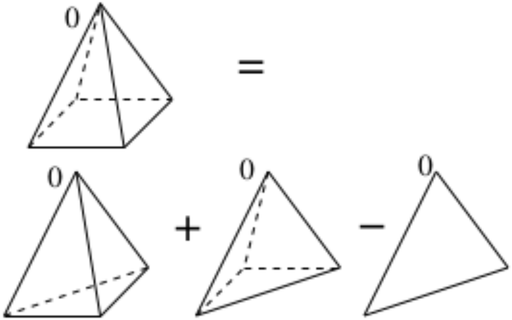

- Alexander Barvinok (1994) [2] proposed an algorithm to decompose any simplicial cones into **unimodular** cones,
- consequently, Barvinok has found the first algorithm to compute $G(P, \mathbf{x})$,
- Moreover, Barvinok's algorithm runs in **polynomial time for a fixed d.** More references on the subject: [1, 10, 14, 19, 22]

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Plan

Sanity-check examples

Example 29 (1)

Input:

$$\{1 \leq i, 1 \leq j, i \leq n, j \leq n\}$$

Output:

$$[[\{n^2\}, [0 \leq n - 1]]]$$

Sanity-check examples

Example 29 (1)

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Example 30 (3)

Input:

$$\{1 \leq i, 1 \leq j, i + j \leq n, 0 \leq n\}$$

Output:

$$[[\{\frac{n^2}{2} - \frac{n}{2}\}, [0 \leq n - 2]]]$$

Examples with several parameters

Example 31 (4)

Input:

$$\{1 \leq i, i \leq n, i \leq m, 1 \leq j, j \leq i\}$$

Output:

$$[[\{1\}, [m-1=0, 0 \leq n-2]],$$
$$[\{\frac{n^2}{2} + \frac{n}{2}\}, [0 \leq m-n, 0 \leq n-1]],$$
$$[\{\frac{m^2}{2} + \frac{m}{2}\}, [0 \leq m-2, 0 \leq n-3, 0 \leq -m+n-1]]]$$

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Example 32 (5)

Input:

$$\{1 \le i, i \le n, i \le m, 1 \le j, j \le p\}$$

Output:

$$\begin{gathered}[[\{pm\}, [n-m \ge 1, p-2 \ge 0, m-1 \ge 0]],\\ [\{pn\}, [m-n \ge 0, n-2 \ge 0, p-1 \ge 0]],\\ [\{1\}, [n-1=0, p-1=0, 0 \le m-1]],\\ [\{p\}, [m-1=0, 0 \le -2+n, 0 \le p-1]]]\end{gathered}$$

Examples with quasi-polynomials

Example 33 (6)

Input:

$$\{1 \le i, j \le n, 2i \le 3j\}$$

Output:

$$[[\{Q([n, 2, [\frac{3n^2}{4} + \frac{n}{2}, -1/4 + \frac{3n^2}{4} + \frac{n}{2}]])\}, [1 \le n]]]$$

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Example 34 (7)

Input:

$$\{0 \leq i, 0 \leq j, j \leq 2i, 2i + j \leq n\}$$

Output:

$$\begin{gathered}[[\{Q([n, 4, [1 + \tfrac{n}{2} + \tfrac{n^2}{8}, 3/8 + \tfrac{n}{2} + \tfrac{n^2}{8}, 1/2 + \tfrac{n}{2} + \tfrac{n^2}{8}, 3/8 + \tfrac{n}{2} + \tfrac{n^2}{8}]])\}, \\ [0 \leq n - 1]], \\ [\{1\}, [n = 0]]]\end{gathered}$$

Integer point counting for parametric polyhedra

Given a parametric polyhedron $P(\vec{b})$, the procedures:

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   a. Putting everything together requires computing with **multivariate quasi-polynomials.**

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8. A **value-constraints** pair is any pair (V, C) where $V \subseteq \mathcal{V}$ and C is a system of constraints.

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6. We assume that we have a procedure that, for any system of constraints C, decides whether C is consistent or not.
7. Then, there exists an algorithm that, for the sequence S computes a non-overlapping sequence T refining S.

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   d. If for $\gamma : p(\mathbf{x}) \geq 0$ of C_1 we have $p(\mathbf{x}) = -1 - u(\mathbf{x})$ and $u(\mathbf{x}) \geq 0$ is a constraint of C_2, then (p, u) is a pair of **adjacent** inequalities.

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5. Theorem: If (p, u) is a pair of adjacent inequalities, and if all other constraints of C_1 (resp. C_2) are valid on C_2 (resp. C_1) then the system of constraints C_3 consisting of all those valid constraints satisfies $Z(C_3) = Z(C_1) \cup Z(C_2)$.

Concluding remarks

Summary and notes

1. We have presented Brion's formula and Barvinok's algorithm for computing the number of integer points of a polytope.
2. We have discussed our adaptation of those works to the case of parametric polyhedra and its implementation in MAPLE.
3. Another adaptation to this parametric case, tailored to compiler optimization, was led by Sven Verdoolaege and is part of a C library called `barvinok`.

Work in progress

1. Our MAPLE implementation aims at supporting Presburger arithmetic
2. This implementation is designed to extend to parametric polyhedra $\mathbf{A}\vec{x} \leq \vec{b}$ where parameters appear not only in $\vec{b}$ but also in $\mathbf{A}$.
3. Our current work focuses on minimizing the number of cases in the discussion and controlling expression swell.

Plan

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Presburger arithmetic (1/2)

Definition 35

The language of **Presburger arithmetic** is:

1. the first-order theory of the integers with addition, equality and order
2. extended by the divisibility predicates $D_k : x \longmapsto k \mid x$, for all $k \in \mathbb{Z}_{>0}$.

For a more formal definition, see Wikipedia's **Presburger arithmetic**.

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Remark 3

A **Presburger formula F in prenex normal form** *has the form:*

$$F = Q_1 x_1 \cdots Q_m x_m \; \phi(x_1, \ldots, x_m, y_1, \ldots, y_n), \tag{6.1}$$

where:

1. $Q_1 x_1 \cdots Q_m x_m$ *is a sequence of quantifiers (existential or universal) and bound variables,*
2. $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is a quantifier-free formula,*
3. $y_1, \ldots, y_n$ *are free (or unbounded) variables.*

Presburger arithmetic (1/2)

Remark 4

1. *We shall assume that any quantifier-free formula $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is in* **disjunctive normal form** *(DNF).*
2. *Hence, it has the form:*

$$\phi(x_1, \ldots, x_m, y_1, \ldots, y_n) \; = \; \bigvee_i \bigwedge_j \; \Phi_{ij}(x_1, \ldots, x_m, y_1, \ldots, y_n), \quad (6.2)$$

   where:

   a. *each $\Phi_{ij}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is an* **atomic formula** *(or* **atom***),*
   b. *thus a formula free of quantifiers and connectives.*

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Remark 5

We can assume that each atom is either

1. *a non-strict inequality $\ell(x_1, \ldots, x_m, y_1, \ldots, y_n) \leq 0$,*
2. *or a divisibility relation $k \mid \ell(x_1, \ldots, x_m, y_1, \ldots, y_n)$,*

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2. *Of course, each variable is meant to take values in $\mathbb{Z}$.*

Quantifier elimination (1/2)

Theorem 36

Presburger arithmetic admits quantifier elimination.

Proof.

1. See the thesis of Mojżesz Presburger [20] the paper of David Cooper [4], and Christoph Haase's **Survival Guide to Presburger Arithmetic** [7].
2. See also our own proof in a few slides.

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Remark 6

Therefore, our goal is to determine the set $D(y_1, \ldots, y_n) \subseteq \mathbb{Z}^n$ *of* **ALL** *integer tuples of* $(y_1, \ldots, y_n)$ *for which the formula* $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is true.*

Quantifier elimination (2/2)

Remark 7

1. *Recall*

$$F = Q_1 x_1 \cdots Q_m x_m \ \phi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

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2. *If $m = 0$, then it "suffices" to determine the tuples of integer values $(y_1, \ldots, y_n)$ for which $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is true.*

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3. *Suppose* $m > 0$*. By induction, assume also* $F = Q x_1 F'$*, where* F' *is quantifier-free.*

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   a. *If $Q = \exists$, then we are now dealing with* **integer projection***, see next section.*
   b. *If $Q = \forall$, then we can replace $\forall x_1 \ F'$ with $\neg(\exists x_1 \ \neg(F'))$,*

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   2. *If $Q = \forall$, then we can replace $\forall x_1 \ F'$ with $\neg(\exists x_1 \ \neg(F'))$,*
   3. *Whenever possible, we should make use of rules like:*

$$\forall x_1 \cdots \forall x_m \ C \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \mathbf{q} \ \Rightarrow \ C = \mathbf{0} \bigwedge \mathbf{q} = \mathbf{0}, \qquad (6.3)$$

   where

      1. *$C \in \mathbb{Z}^{r \times m}$ is a matrix, and*

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   2. *$\mathbf{q} \in (\mathbb{Z}[y_1, \ldots, y_m])^r$ is a vector of linear polynomials.*

Coarsening the atoms

Remark 8

In Cooper's algorithm, when processing $\exists x_1\ F'$, the formula F' uses the following four types of atoms:

$$A_y < ax_1,\ ax_1 < A_y,\ k \mid (ax_1 + A_y),\quad \text{and}\quad \neg (k \mid (ax_1 + A_y))\,, \tag{6.4}$$

where $a \in \mathbb{Z}$ and $A_y \in \mathbb{Z}[y_1, \ldots, y_n]$ is a linear polynomial.

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where $a \in \mathbb{Z}$ and $A_y \in \mathbb{Z}[y_1, \ldots, y_n]$ is a linear polynomial.

Remark 9

We can rearrange our quantifier-free formula to:

$$\phi(x_1, \ldots, x_m, y_1, \ldots, y_n) \ = \ \bigvee_i \ Z_i(x_1, \ldots, x_m, y_1, \ldots, y_n), \qquad (6.5)$$

where each Z_i is a predicate of the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_m \end{pmatrix} \in \ \mathbb{Z}\text{Polyhedron}(P_i, L_i), \qquad (6.6)$$

for some polyhedra P_i and integer lattices L_i.
We call such a predicate a **$\mathbb{Z}$-polyhedron predicate**.

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Theorem 37 (parametric multivariate CRT)

The values of $(w_1, \ldots, w_\nu)$ for which the above system has solutions form a lattice of $\mathbb{Z}^\nu$. Moreover, for each value of $(w_1, \ldots, w_\nu)$, the $\mathbf{z}$*-solutions form a lattice of $\mathbb{Z}^n$.*

Proof.

Compute the Hermite normal forms of the appropriate matrices. □

Plan

Integer projection: $n = 1$

Remark 10

1. *From the above section, we consider the formula*

$$\exists x \; \phi(x, y_1, \ldots, y_n), \quad \text{where} \quad \phi(x, y_1, \ldots, y_n) = \bigvee_i \phi_i(x, y_1, \ldots, y_n), \tag{6.8}$$

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Remark 11

1. *Let $f_1, \ldots, f_s$, $g_1, \ldots, g_r \in \mathbb{Z}[x, \mathbf{y}]$ be linear and let $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$.*

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$$\exists x\ \phi(x, y_1, \ldots, y_n), \quad \text{where} \quad \phi(x, y_1, \ldots, y_n) = \bigvee_i \ \phi_i(x, y_1, \ldots, y_n), \tag{6.8}$$
where $\phi_i(x, y_1, \ldots, y_n)$ *is a conjunction of congruence relations and non-strict inequalities.*
2. *We want the values of* $\mathbf{y} = y_1, \ldots y_n$ *so that* $\exists x\ \phi(x, y_1, \ldots, y_n)$ *holds.*
3. *We can further reduce the problem as follows.*

Remark 11

1. *Let* $f_1, \ldots, f_s,\ g_1, \ldots, g_r \in \mathbb{Z}[x, \mathbf{y}]$ *be linear and let* $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$.
2. *Consider the formula:*
$$F(\mathbf{y}) : (\exists x \in \mathbb{Z}) \left\{ \begin{array}{ccc} f_1 & \leq & 0 \\ \vdots & \vdots & \vdots \\ f_s & \leq & 0 \end{array} \right. \wedge \begin{array}{cccc} g_1 & \equiv & 0 & \bmod k_1 \\ \vdots & \vdots & \vdots & \\ g_r & \equiv & 0 & \bmod k_r \end{array} \tag{6.9}$$

Integer projection: $n = 1$

Remark 10

1. *From the above section, we consider the formula*
$$\exists x\ \phi(x, y_1, \ldots, y_n), \quad \text{where} \quad \phi(x, y_1, \ldots, y_n) = \bigvee_i \phi_i(x, y_1, \ldots, y_n), \tag{6.8}$$
where $\phi_i(x, y_1, \ldots, y_n)$ is a conjunction of congruence relations and non-strict inequalities.
2. *We want the values of $\mathbf{y} = y_1, \ldots y_n$ so that $\exists x\ \phi(x, y_1, \ldots, y_n)$ holds.*
3. *We can further reduce the problem as follows.*

Remark 11

1. *Let $f_1, \ldots, f_s$, $g_1, \ldots, g_r \in \mathbb{Z}[x, \mathbf{y}]$ be linear and let $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$.*
2. *Consider the formula:*
$$F(\mathbf{y}) : (\exists x \in \mathbb{Z}) \left\{ \begin{array}{ccc} f_1 & \leq & 0 \\ \vdots & \vdots & \vdots \\ f_s & \leq & 0 \end{array} \right. \wedge \begin{array}{cccc} g_1 & \equiv & 0 & \bmod k_1 \\ \vdots & \vdots & \vdots & \\ g_r & \equiv & 0 & \bmod k_r \end{array} \tag{6.9}$$
3. *We shall determine the set $D(\mathbf{y})$ of integer tuples $(y_1, \ldots y_n)$ for which $F(\mathbf{y})$ holds. We call $D(\mathbf{y})$ the* **integer projection** *of $F(\mathbf{y})$.*

Integer projection: $n = 1$, removing congruences

Remark 12

1. *Suppose that $r > 0$ holds, that is, we do have congruences.*

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7. *If that process* **solves for** x*, then our problem becomes that of describing the points of a $\mathbb{Z}$-polyhedron, which can be done by our integer hull algorithm.*
8. *If that process* **does not solve for** x*, then go to next slide.*

Integer projection: $n = 1$, no congruences, 2 inequalities

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2. *We start with the case $s = 2$ and rename f_1, f_2 to A, B.*
3. *We also write:*

$$A = A_{\mathbf{y}} - a\,x, \quad \text{and} \quad B = -B_{\mathbf{y}} + b\,x, \tag{6.10}$$

where $a, b \in \mathbb{Z}$ are non-zero and where $A_{\mathbf{y}}, B_{\mathbf{y}} \in \mathbb{Z}[\mathbf{y}]$ are linear

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5. *With these assumptions, we call the inequalities* $A \leq 0$ *and* $B \leq 0$, *respectively a* **lower bound** *and an* **upper bound** *for* x.
6. *Observe that Formula (6.9* null*) simplifies to:*
$$F(\mathbf{y}) : (\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \leq a\,x)\ \wedge\ (b\,x \leq B_{\mathbf{y}}), \tag{6.11}$$

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7. *We present a first formula for $D(\mathbf{y})$ based on Harris Williams [23, 24]*

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7. *We present a first formula for $D(\mathbf{y})$ based on Harris Williams [23, 24]*
8. *Then, we present a second one based on William Pugh's Omega test [16, 17].*

Williams-style Projection

Theorem 38

Let $\ell = \mathrm{lcm}(a, b)$*,* $b' = \ell/a$ *and* $a' = \ell/b$*. For* $0 \le k < b$*, define*

$$E_k := \{\mathbf{y} \mid \mathrm{rem}(B_{\mathbf{y}}, b) = k\}.$$

Then, the following two formulas are equivalent:

1. $F(\mathbf{y})$*:* $(\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \le a\,x) \ \wedge\ (b\,x \le B_{\mathbf{y}})$,
2. $\bigvee_{k=0}^{k=b-1} (\mathbf{y} \in E_k) \wedge (a'k \ \le\ a' B_{\mathbf{y}} - b' A_{\mathbf{y}})$.

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Proof.

1. If $a = 1$ or $b = 1$ holds, then: $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \le a'\,B_{\mathbf{y}}$.

□

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Proof.

1. If $a = 1$ or $b = 1$ holds, then: $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \le a'\,B_{\mathbf{y}}$.
2. From now on, assume $a > 1$ and $b > 1$ both hold. Observe that

$$F(\mathbf{y}) \iff (\exists x \in \mathbb{Z})\ (b' A_{\mathbf{y}} \le \ell\,x)\ \wedge\ (\ell\,x \le a' B_{\mathbf{y}}).$$

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3. Hence, $F(\mathbf{y})$ says that a multiple of ℓ lies between $b'A_{\mathbf{y}}$ and $a'B_{\mathbf{y}}$).

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Then, the following two formulas are equivalent:

1. *F(y):* $(\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \le a\,x)\ \wedge\ (b\,x \le B_{\mathbf{y}})$,
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$$F(\mathbf{y}) \iff (\exists x \in \mathbb{Z})\ (b'A_{\mathbf{y}} \le \ell\,x)\ \wedge\ (\ell\,x \le a'B_{\mathbf{y}}).$$
3. Hence, $F(\mathbf{y})$ says that a multiple of ℓ lies between $b'A_{\mathbf{y}}$ and $a'B_{\mathbf{y}}$).
4. Thus, $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \ \le\ a'\,B_{\mathbf{y}} - \operatorname{rem}(a'\,B_{\mathbf{y}}, \ell)$.
5. That is, $F(\mathbf{y}) \iff b'A_{\mathbf{y}} \ \le\ a'(B_{\mathbf{y}} - \operatorname{rem}(B_{\mathbf{y}}, b))$.

□

Pugh's omega test (1/2)

Lemma 39 (William Pugh)

If we have:

$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \ \geq \ (a-1)(b-1) \tag{6.12}$$

then $F(\mathbf{y})$ *holds.*

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1. Consider the closed interval: $I \;:=\; \left(\frac{A_{\mathbf{y}}}{a}, \frac{B_{\mathbf{y}}}{b}\right)$.

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Proof.

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2. If I does not contain an integer, then we have:

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3. Let $\rho := \mathrm{rem}(A_{\mathbf{y}}, a)$. Since $i < \frac{A_{\mathbf{y}}}{a}$ holds, we have:
$$A_{\mathbf{y}} \ = \ i\,a + \rho \ \ \text{and} \ 0 < \rho < a, \tag{6.14}$$
4. from which we deduce: $\frac{A_{\mathbf{y}}}{a} - i \ \geq \ \frac{1}{a}$.

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3. Let $\rho := \mathrm{rem}(A_{\mathbf{y}}, a)$. Since $i < \frac{A_{\mathbf{y}}}{a}$ holds, we have:
$$A_{\mathbf{y}} \ = \ i\,a + \rho \ \text{ and } 0 < \rho < a, \tag{6.14}$$
4. from which we deduce: $\frac{A_{\mathbf{y}}}{a} - i \ \geq \ \frac{1}{a}$.
5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \ \geq \ \frac{1}{b}$.

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3. Let $\rho := \mathrm{rem}(A_{\mathbf{y}}, a)$. Since $i < \frac{A_{\mathbf{y}}}{a}$ holds, we have:
$$A_{\mathbf{y}} \ = \ i\,a + \rho \ \text{ and } 0 < \rho < a, \tag{6.14}$$
4. from which we deduce: $\frac{A_{\mathbf{y}}}{a} - i \ \geq \ \frac{1}{a}$.
5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \ \geq \ \frac{1}{b}$.
6. From the above two inequalities, elementary manipulations yield:
$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \ \leq \ ab - a - b. \tag{6.15}$$

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3. Let $\rho := \mathrm{rem}(A_{\mathbf{y}}, a)$. Since $i < \frac{A_{\mathbf{y}}}{a}$ holds, we have:
$$A_{\mathbf{y}} \;=\; i\,a + \rho \;\; \text{and} \; 0 < \rho < a, \qquad (6.14)$$
4. from which we deduce: $\frac{A_{\mathbf{y}}}{a} - i \;\geq\; \frac{1}{a}$.
5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \;\geq\; \frac{1}{b}$.
6. From the above two inequalities, elementary manipulations yield:
$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \;\leq\; ab - a - b. \qquad (6.15)$$
7. Therefore, if the above inequality does not hold, that is, if $aB_{\mathbf{y}} - bA_{\mathbf{y}} \;\geq\; (a-1)(b-1)$ does hold, then I contains an integer.

Pugh's omega test (2/2)

Theorem 40

Define $\kappa(a,b) := \lceil \frac{(a-1)(b-1)}{a'} \rceil$. *Then, Formula* $F(\mathbf{y})$ *is equivalent to:*

$$((a-1)(b-1) \leq aB_{\mathbf{y}} - bA_{\mathbf{y}}) \bigvee_{k=\kappa(a,b)}^{k=b-1} (\mathbf{y} \in E_k) \wedge (a'k \leq a'B_{\mathbf{y}} - b'A_{\mathbf{y}}) \,. \tag{6.16}$$

Proof.

This is a direct consequence of William Pugh's lemma and Harris Williams' projection formula □

Remark 14

1. *William Pugh's lemma reduces significantly the number of "cuts"*
2. *To take a concrete example, say with* $a = 7$ *and* $b = 11$*:*
   - a. *with Williams' projection alone* k *ranges from* 0 *to* 10*,*
   - b. *with William Pugh's lemma,* k *ranges from* 8 *to* 10*.*

Integer projection: $n = 1$, s inequalities

We now describe a procedure Projection$(f_1, \ldots, f_s; x)$ computing $D(\mathbf{y})$.

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We now describe a procedure Projection$(f_1, \ldots, f_s; x)$ computing $D(\mathbf{y})$.

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We now describe a procedure Projection($f_1, \ldots, f_s; x$) computing $D(\mathbf{y})$.

1. If $f_1, \ldots, f_s$ only count lower (resp. upper) bounds for x, then return true.
2. Initialize $D(\mathbf{y})$ to true.
3. For each pair (A, B) consisting of a lower bound and an upper bound of x, replace $D(\mathbf{y})$ with $D(\mathbf{y}) \wedge$ Projection(A, B), where Projection(A, B) is given by Pugh's omega test.

Integer projection: $n = 1$, s inequalities

We now describe a procedure Projection$(f_1, \ldots, f_s; x)$ computing $D(\mathbf{y})$.

1. If $f_1, \ldots, f_s$ only count lower (resp. upper) bounds for x, then return true.
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   c. S_0 is the conjunction of the $((a-1)(b-1) \leq aB_{\mathbf{y}} - bA_{\mathbf{y}})$, for all pairs (A, B) of lower and upper bounds of x.

Plan

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