Triangular Decomposition of Polynomial Systems: Algorithmic Advances and Remaining Challenges

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ICMM'09
In honor of Prof. Wen-Tsün Wu
Beijing, China,
May 13, 2009

Wu's Characteristic Set Method

Input: $F \subset \mathbf{k}[x_1, \ldots, x_n]$ and a **variable ordering** $\leq$.

Output: C a Wu characteristic set of F.

repeat

(S) $B := \mathrm{MinimalAutoreducedSubset}(F, \leq)$

(R) $A := F \setminus B$;

$R :=$ **prem** (A, B)

(U) $R := R \setminus \{0\}$; $F := F \cup R$

until $R = \emptyset$

return B

- Repeated calls decomposes $V(F)$ into quasi-components.

Buchberger's Algorithm

Input: $F \subset \mathbf{k}[x_1, \ldots, x_n]$ and a **term order** $\leq$.

Output: G a reduced Gröbner basis w.r.t. $\leq$ of the ideal $\langle F \rangle$ generated by F.

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(S) $B := \mathrm{MinimalAutoreducedSubset}(F, \leq)$
(R) $A :=$ **S_Polynomials** $(B) \cup F$;
$R :=$ **Reduce** $(A, B, \leq)$
(U) $R := R \setminus \{0\}$; $F := F \cup R$
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The Notion of a Regular Chain (1/2)

- Let $T \subset \mathbf{k}[x_1 < \cdots < x_n] \setminus \mathbf{k}$ be a triangular set, hence the polynomials of T have pairwise distinct main variables.

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- Let $\mathrm{mvar}(T) := \{\mathrm{mvar}(t) \ \mid \ t \in T\}$, $\mathrm{init}(t) := \mathrm{lc}(t, \mathrm{mvar}(t))$ for all $t \in T$, and $h_T := \prod_{t \in T} \mathrm{init}(t)$.

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- T is a regular chain if for each $v \in \mathrm{mvar}(T)$ the initial of T_v is regular modulo $\mathrm{sat}(T_{<v})$ (Michael Kalkbrener 91).

The Notion of a Regular Chain (2/2)

- Let $p \in \mathbf{k}[x_1 < \cdots < x_n]$ and $T := T_{<w} \cup T_w \subset \mathbf{k}[x_1 < \cdots < x_n]$ be a triangular set. The *iterated resultant* of p w.r.t. T is:

$$\mathrm{res}(p, T) \quad = \quad \begin{cases} p & \text{if } \deg(f, w) = 0 \\ \mathrm{res}(\mathrm{res}(p, T_w, w), T_{<w}) & \text{otherwise} \end{cases}$$

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$$\{p \mid \mathrm{prem}(p, T) = 0\} \quad = \quad \mathrm{sat}(T)$$

(Philippe Aubry, Daniel Lazard, Marc Moreno Maza 97).

Regular Chain under Specialization

- Let $\mathbf{u} = u_1, \ldots, u_d$ be parameters, $\mathbf{y} = y_1, \ldots, y_m$ be unknowns, Π_U be the projection from $\mathbf{K}^{m+d}$ to $\mathbf{K}^d$.

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- A regular chain $T \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ *specializes well* at $u \in \mathbf{K}^d$ if $T(u)$ is a regular chain in $\mathbf{K}[\mathbf{y}]$ and $\mathrm{rank}(T(u)) = \mathrm{rank}(T_{>U_d})$.

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- Let $T \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ be a reg. chain and $u \in \Pi_U(\mathbf{W}(T \cap \mathbf{k}[U]))$. T specializes well at $u \iff \mathrm{res}(h_{T_{>U_d}}, T_{>U_d}) \neq 0$ at $\mathbf{u} = u$

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- Replacing *regular chain* by squarefree reg. ch. in char. 0 and $h_{T_{>U_d}}$ by $\mathrm{Sep}_{T_{>U_d}}$ one obtains the *border polynomial of T*.

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Related Work

On a projection theorem of quasi-varieties in elimination theory (Wen-Tsün Wu 90). (Xiao-Shan Gao, Shang-Ching Chou 92) (Dongming Wang 00 & 01) (Lu Yang, Xiaorong Hou, Bican Xia 01) (Xiao-Shan Gao, Ding-Kang Wang 03) (Changbo Chen, Oleg Golubitsky, François Lemaire, Marc Moreno Maza, Wei Pan 07)

Incremental Solving

- Let $F \subset \mathbf{k}[\mathbf{x}]$, $f \in \mathbf{k}[\mathbf{x}]$, $T, T^m \ldots, T^e \subset \mathbf{k}[\mathbf{x}]$ reg. chains. Assume we have *solved* F as $V(F) = W(T^i) \cup \cdots \cup W(T^e)$.

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- Assume that we have an operation $(f, T) \longmapsto \mathbf{Intersect}(f, T) = (C_1, \ldots, C_d)$ such that

$$V(f) \cap W(T) \ \subseteq \ \cup_i W(C_i) \ \subseteq \ V(f) \cap \overline{W(T)}.$$

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Related Work

(D. Lazard 91) proposes the principle. (M. M. M. 00) introduces regular GCDs and gives a complete incremental algorithm which, in addition, generates components by decreasing order of dimension.

The notion of a Regular GCD

- Let $P, Q, G \in \mathbf{k}[x_1 < \cdots < x_n][y]$ and $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ reg. chain. G is a *regular GCD* of P, Q modulo $\mathrm{sat}(T)$ if
  - *(i)* $\mathrm{lc}(G, y)$ is a regular modulo $\mathrm{sat}(T)$,
  - *(ii)* $G \in \langle P, Q \rangle$ modulo $\mathrm{sat}(T)$,
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- If both $T \cup P$ and $T \cup Q$ are regular chains and if G is a GCD of P, Q modulo $\mathrm{sat}(T)$ with $\deg_y(G) > 0$ then we have

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Related Work

(M. Kalkbrener 91) (M. M. M., Renaud Rioboo 95) (M. M. M. 00)

Regular GCDs: Bottom-up or Top-down?

- Let $P, Q \in \mathbf{k}[x_1 < \cdots < x_n][y]$ and $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ reg. chain. How to compute a *regular GCD* of P, Q mod $\mathrm{sat}(T)$?
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- (M. M. M. and R. Rioboo 95) assume $\mathrm{sat}(T)$ radical $+$ 0-dimensional and use the subresultant chain of P, Q directly in $\mathbf{k}[x_1 < \cdots < x_n][y]$ mod $\mathrm{sat}(T)$. Better but removing the assumptions removes the efficiency.

Equiprojectable Decomposition (1/2)

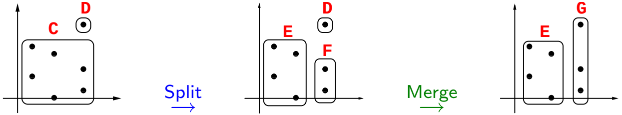
$$C \left| \begin{array}{l} C_2 = y^2 + 6yx^2 + 2y + x \\ C_1 = x^3 + 6x^2 + 5x + 2 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$\downarrow$ Split C : GCD $\downarrow$

$$E \left| \begin{array}{l} {C_2}' = y^2 + x \\ {C_1}' = x^2 + 5 \end{array} \right. , \quad F \left| \begin{array}{l} C_2'' = y^2 + y + 1 \\ C_1'' = x + 6 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$\downarrow$ Merge F and D : CRT $\downarrow$

$$E \left| \begin{array}{l} C_2' = y^2 + x \\ C_1' = x^2 + 5 \end{array} \right. , \quad G \left| \begin{array}{l} G_2 = y^3 + 6 \\ G_1 = x + 6 \end{array} \right.$$



Equiprojectable Decomposition (2/2)

$$
\begin{tikzcd}
\begin{array}{c} F \subset \mathbb{Q}[x_1, \ \ldots, \ x_n] \\ p \text{ prime, } p > A(F) \end{array} \arrow[r] \arrow[d, "\text{Triang.}(F \mod p)"'] & Eq.D._{\mathbb{Q}}(F) \\
\begin{array}{c} \{T^1, \ldots, \ T^s\} \\ \subset \mathbb{F}_p[x_1, \ \ldots, \ x_n] \end{array} \arrow[r, "\textbf{Split + Merge}"] & Eq.D._{\mathbb{F}_p}(F) \arrow[u, "\textit{Lifting}"']
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- $A(F) := 2n^2 d^{2n+1}(3h + 7\log(n+1) + 5n \log d + 10)$ where h and d upper bound coeff. sizes and total degrees for $f \in F$. Assumes F square and generates a 0-dimensional radical ideal.

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- If $p \nmid A(F)$, the equiprojectable decomposition specializes well mod p.

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- If $p \not| A(F)$, the equiprojectable decomposition specializes well mod p.
- In practice we choose p much smaller with a probability of success, i.e. $> 99\%$ with $p \approx ln(A(F))$ (Xavier Dahan, M. M. M., Éric Schost, Wenyuan Wu, Yuzhen Xie 05).

Fast Polynomial Arithmetic (1/2)

- Let A and B in $\mathbf{k}[x_1, \ldots, x_n]$ reduced w.r.t. $T := \{T_1, \ldots, T_n\}$ 0-dimensional reg. chain with all $\mathrm{init}(T_i) = 1$.
- The size of input is $\delta_{\mathbf{T}} = \deg(T_1, x_1) \cdots \deg(T_n, x_n)$.

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- One can compute $A\,B \mod \langle T_1, \ldots, T_n\}$ in $O\tilde{}(4^n \delta_{\mathbf{T}})$ operations in $\mathbf{k}$ (Xin Li, M.M.M., É. Schost 07).

Fast Polynomial Arithmetic (1/2)

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- Two key ideas: using the fast division trick of (Sieveking 72) (Kung 74) and avoid $\mod \langle T_1, \ldots, T_n\}$ as much as possible.

$\underline{\mathrm{ModMul}(A, B, \{T_1, \ldots, T_n\})}$

1 $D := AB$ computed in $\mathbf{k}[x_1, \ldots, x_n]$

2 **return** $\mathrm{NormalForm}_n(D, \{T_1, \ldots, T_n\})$

Fast Polynomial Arithmetic (2/2)

$\mathsf{NormalForm}_1(A : R[x_1], \{T_1 : R[x_1]\})$

$S_1 := \mathrm{Rev}(T_1)^{-1} \mod x_1^{\deg(A)-\deg(T_1)+1}$

$D := \mathrm{Rev}(A) S_1 \mod x_1^{\deg(A)-\deg(T_1)+1}$

$D := T_1 \, \mathrm{Rev}(D)$

return $A - D$

$\mathsf{NormalForm}_2(A : R[x_1, x_2], \{T_1 : R[x_1], T_2 : R[x_1, x_2]\})$

$A := \mathrm{map}(\mathsf{NormalForm}_1, \mathrm{Coeffs}(A, x_2), \{T_1\})$

$S_2 := \mathrm{Rev}(T_2)^{-1} \mod T_1, x_2^{\deg(A,x_2)-\deg(T_2,x_2)+1}$

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Subresultants and Regular GCDs

- Let $P, Q \in \mathbf{k}[x_1 < \cdots < x_n][y]$ with main variable y.
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- Assume
  - $\mathrm{res}(P, Q, y) \in \mathrm{sat}(T)$,
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  - Let $1 \leq d \leq \deg(Q, y)$ such that $S_j \in \mathrm{sat}(T)$ for all $0 \leq j < d$.
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Theorem (Xin Li, M.M.M., Wei Pan 2009)

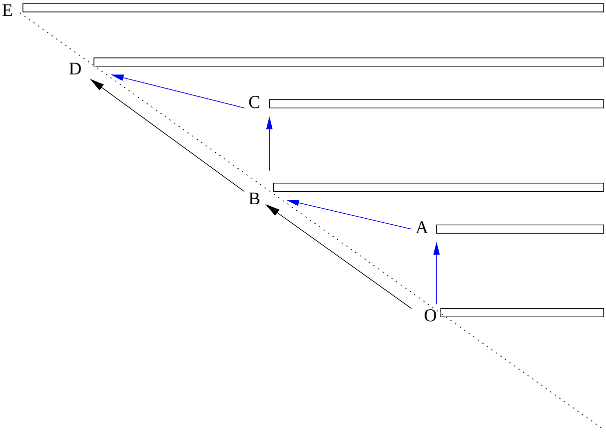
Assume that one of the following conditions holds:

- $\mathrm{sat}(T)$ *is radical,*
- *for all* $d < k \le \mathrm{mdeg}(Q)$, *the coefficient of* y^k *in* S_k *is either null or regular modulo* $\mathrm{sat}(T)$.

Then, S_d *is a regular GCD of* P, Q *modulo* $\mathrm{sat}(T)$.

Computing Regular GCDs: Bottom-up!

- Assume that the subresultants S_j for $1 \leq j < \mathrm{mdeg}(Q)$ are computed.
- Then one can compute a regular GCD of P, Q modulo $\mathrm{sat}(T)$ by performing a bottom-up search.



Computing Regular GCDs: Complexity Estimates!

- Let $x_{n+1} := y$. Define $d_i := \max(\deg(P, x_i), \deg(Q, x_i))$. Define $b_i := 2 d_i d_{n+1}$ and $B := (b_1 + 1) \cdots (b_n + 1)$.

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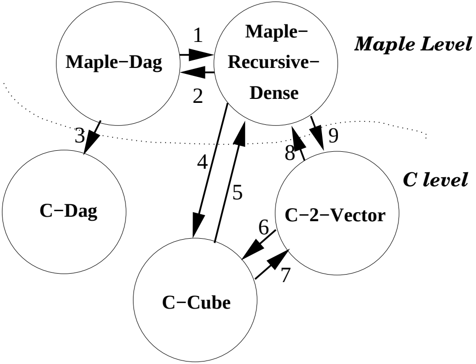
- Best known results for practical sizes, say $d_{n+1} < 500$.
- If a regular GCD is expected to have degree 1 in y all computations fit in $O\tilde{}(d_{n+1} B)$ which is essentially optimal.

The RegularChains library in MAPLE

- 80,000 lignes of MAPLE code, 36,000 lignes of C code, 121 Commands, 6 modules ChainTools, MatrixTools, ConstructibleSetTools. ParametricSystemTools, SemiAlgebraicSetTools , FastArithmeticTools .
- Main new commnands in MAPLE 13: IsPrimitive, ComplexRootClassification, RealRootClassification, RealRootIsolate RealRootCounting, BorderPolynomial, + those of FastArithmeticTools (see demo).
- Current contributors: Changbo Chen, Francçois Lemaire, Liyun Li, Xin Li, M.M.M., Wei Pan, Bican Xia, Rong Xiao, Yuzhen Xie.

The MODPN library

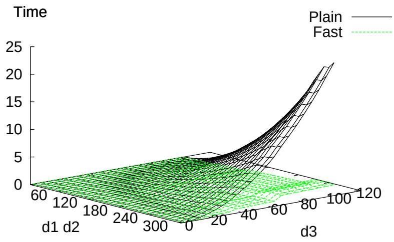


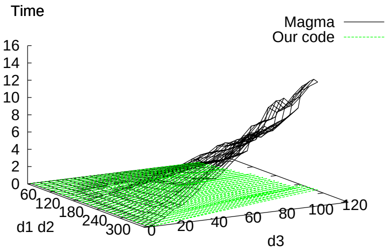
- *C-Dag* for straight-line program.
- *C-Cube* for FFT-based computations.
- *C-2-Vector* for compact dense representation.
- *Maple-Dag* for calling `RegularChains` library.
- *Maple-Recursive-Dense* for calling RECDEN library.

Fast Normal Form Benchmarks

[left] comparison of classical (plain) and asymptotically fast strategies.

[right] comparison with MAGMA.

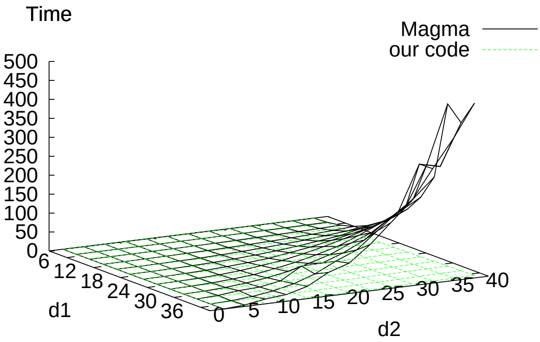




- Asymptotically fast strategy dominates the classical one.
- Our fast implementation is better than Magma's one (the best known implementation).

Generic Bivariate Systems

- "our code" means `BivariateModularTriangularize` in MAPLE 13.
- Random generic input systems, thus equiprojectable.
- For the largest examples (having about 5700 solutions), the ratio is about 460/7 in our favor.



Non-generic Bivariate Systems

- Examples designed to enforce many "splittings" (many equiprojectable components).
- For the largest examples, the ratio is 5260/80, in our favor.

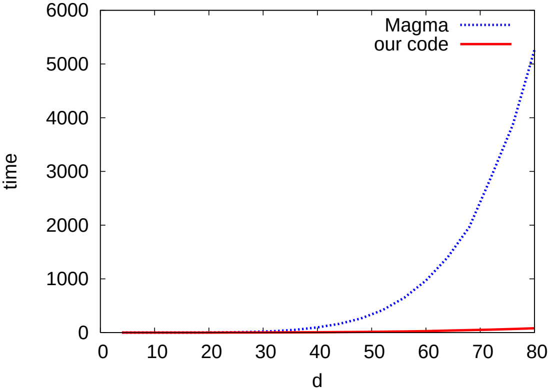


Figure: Non-generic bivariate systems: MAGMA vs. us.

Generic Trivariate Systems

- MAPLE means the experimental and fast version of `Triangularize` to be integrated in MAPLE 14.

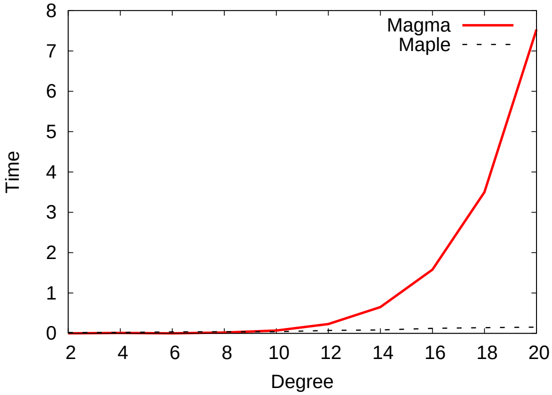


Figure: *Generic dense 3-variable.*

Regularity Test ($=$ Saturation)

d_1	d_2	d_3	Regularize	Fast Regularize	Magma
2	2	3	0.032	0.004	0.010
3	4	6	0.160	0.016	0.020
4	6	9	0.404	0.024	0.060
5	8	12	$>$100	0.129	0.330
6	10	15	$>$100	0.272	1.300
7	12	18	$>$100	0.704	5.100
8	14	21	$>$100	1.276	14.530
9	16	24	$>$100	5.836	40.770
10	18	27	$>$100	9.332	107.280
11	20	30	$>$100	15.904	229.950
12	22	33	$>$100	33.146	493.490

Table: Generic dense 3-variable.

- In the non-generic case, both gaps are even larger.
- "Fast Regularize" means `RegularizeDim0` in MAPLE 13.

Do Triangular Decompositions Have the Rigth Size?

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- Let $T \subset \mathbf{k}[x_1, \ldots, x_n]$ be a 0-dimensional reg. chain with all $\mathrm{init}(T_i) = 1$. There exists a 0-dimensional reg. chain N such that $V(T) = V(N)$ and the height (or "size") of each coefficient in N is upper bounded by
  - the height of $\mathbf{V}(T)$, that is, the minimum size of a data set encoding $\mathbf{V}(T)$, if $\mathbf{k} = \mathbb{Q}$,
  - the degree of $\mathbf{V}(T^{\downarrow})$, if $\mathbf{k}$ is a field $\mathbb{F}_p(t_1, \ldots, t_m)$ of rational functions and $T^{\downarrow}$ is T regarded in $\mathbf{k}[t_1, \ldots, t_m, x_1, \ldots, x_n]$.

  See (X. Dahan, É. Schost 04) for precise statements.

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- Let $\mathcal{I} \subset \mathbf{k}[x_1 < x_2]$ be 0-dimensional radical with degree d and h be the height of $V(\mathcal{I})$. There exists a (non-reduced) lexicographical Gröbner basis G of $\mathcal{I}$ such that the height of a coefficient is essentially quadratic in both h and d. These estimatas are sharp. Estimates become cubic for the reduced basis. (X. Dahan 09).

What Plays the Role of Degree Bases for Triangular Decompositions ?

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- In my opinionm Triangular Decompositions themselves do:
  - Even if degree bases instead of lex bases are used in the previous benchmarks, the bivariate and triavariate `Triangularize` solvers remain faster.
  - Keep in mind that a triangular decomposition is "essentially" a factored lex basis (D. Lazard 92)
  - For Generic input system $\mathbb{F}_p[x_1, \ldots, x_n]$ in degree $d = 2$,

$n = 3$ the output of `Triangularize` vs that of `Basis` gies 7200 vs 13400 characters long

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- Size estimates are promising but not enough. More on this another time.

Can Triangular Decomposition Preserve Multiplicities?

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- See previous talk.

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- For overconstrained systems one may conider restarting from the elimnation methods (Wen-Tsün Wu, Dongming Wang, Kalkbrener, ...) making use of modular methods, fast arithmetic, multivariate resultants, etc.

Can we unify terminology?

Xie Xie! Thank You!