# Osculating spaces of plane analytic curves

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Note that $a_{00} = 0$ and $F$ has $N_d$ unknown coefficients.

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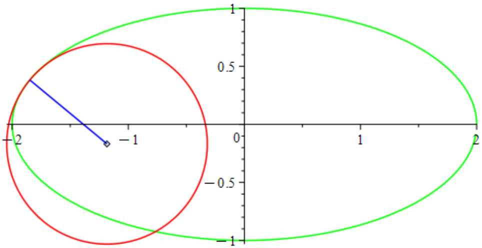

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4. Osculating curves have applications in CAD (Computer-Aided Design), CNC (Computer Numerically Controlled) machines and 3D Printing.

# Contact Order

## Definition

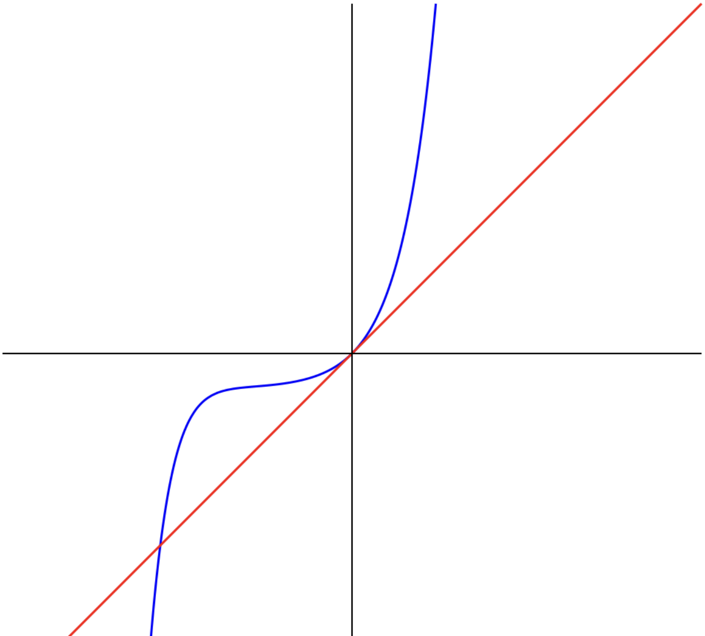
We say that the curve $F(x, y)$ has *contact order* $k$ with the graph $\Gamma$ of $x \mapsto f(x)$ at the origin if

$$F(x, f(x)) \equiv 0 \bmod x^j \quad \text{for } j = 1, \ldots, k$$

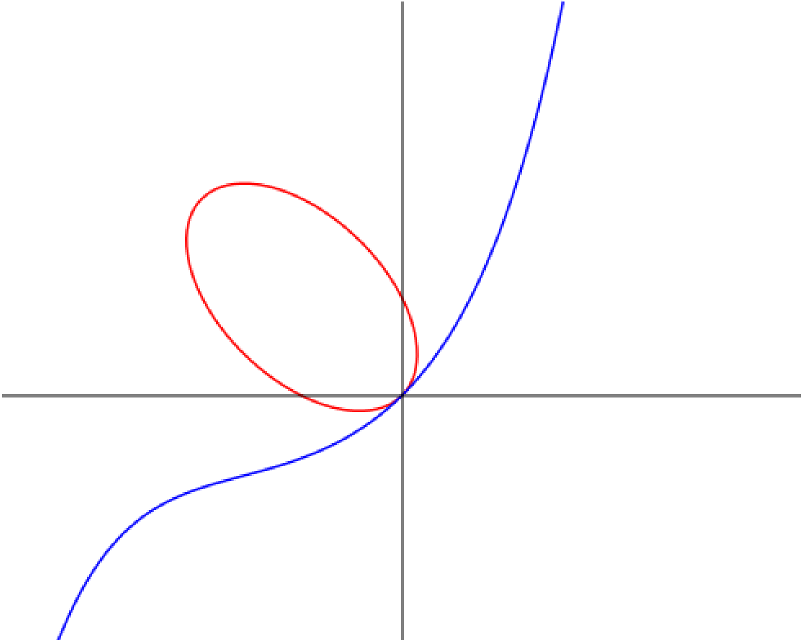
$$F(x, f(x)) \not\equiv 0 \bmod x^{k+1}$$

This means that the function $F(x, f(x))$ vanishes up to order $k$ at $x = 0$.

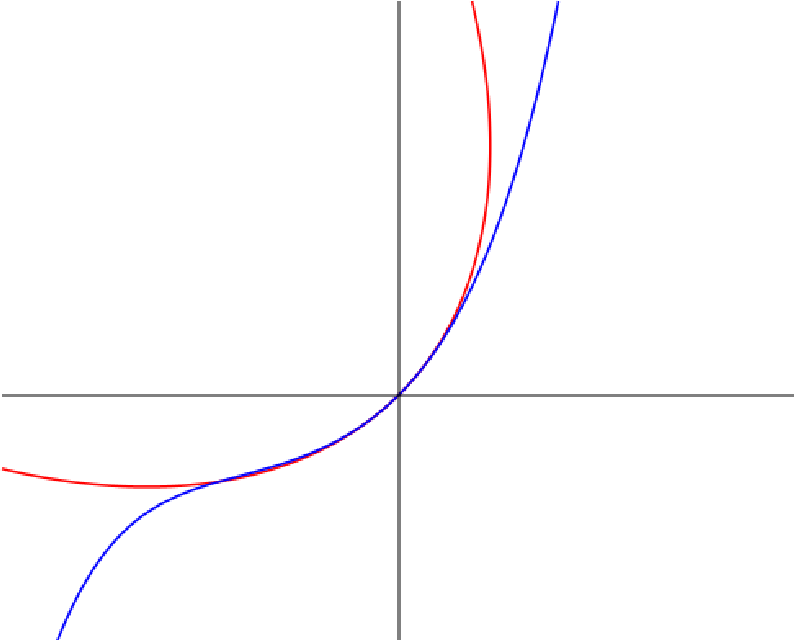
# Approximant of $f = x + x^2 + x^3 + x^4 + x^5$ of order 1



# Approximant of $x + x^2 + x^3 + x^4 + x^5$ of order 2



# Approximant of $x + x^2 + x^3 + x^4 + x^5$ of order 3



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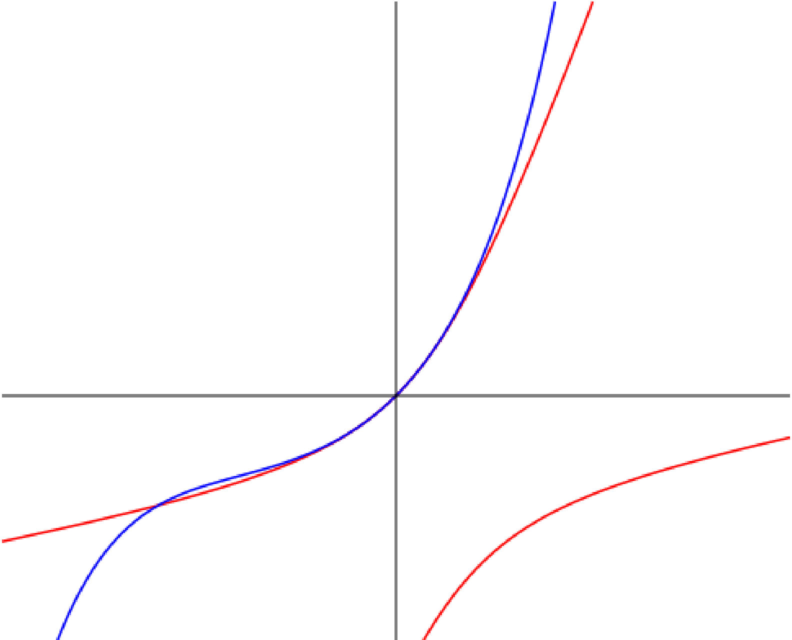
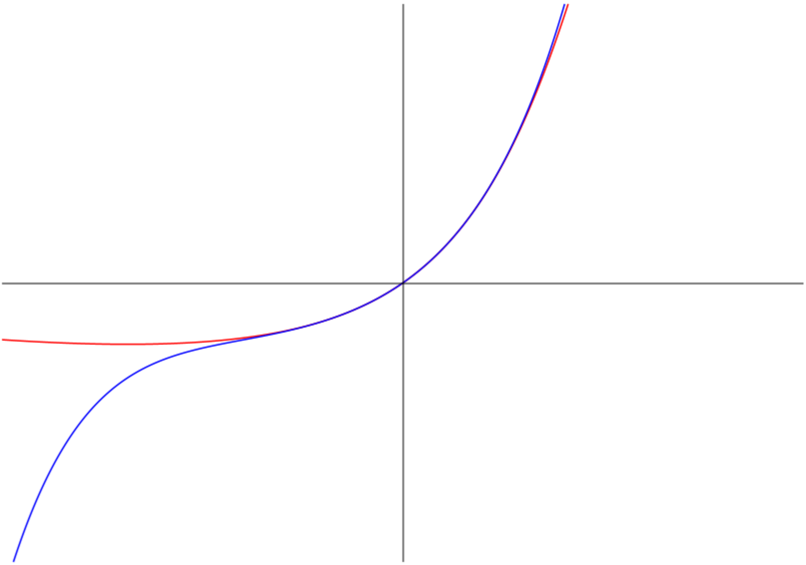


# Approximant of $x + x^2 + x^3 + x^4 + x^5$ of order 5



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3. Denote by $h_k(\mathbf{a}, \mathbf{c})$ the coefficient of $x^k$ in $F(x, f(x))$, that is,

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## Coefficients of the linear system $h_k(\mathbf{a}, \mathbf{c}) = 0$

1. For $j, \delta, n \in \mathbb{N}$, let $R^n_{j,\delta}$ be all $(n)$-tuples of natural numbers $\mathbf{r} = (r_1, \ldots, r_n)$ such that $\sum_{t=1}^n r_t = j$ and $\sum_{t=1}^n t \cdot r_t = \delta$.

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2. Writing $\mathbf{c} = (c_1, \ldots, c_n)$, define

$$q_{j,\delta}(c_1, \ldots, c_n) = \sum_{\mathbf{r} \in R^n_{j,\delta}} \mathbf{c}^{\mathbf{r}},$$

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3. Applying Faà di Bruno's formula, we have:

$$\frac{\partial h_k}{\partial a_{ij}} = \sum_{\mathbf{r} \in R^{N_d - 1}_{j,k-i}} \mathbf{c}^{\mathbf{r}} = q_{j,k-i}(c_1, \ldots, c_{N_d - 1}).$$

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3. Substituting gives:

$$\begin{aligned}
a_{00} &= [x^0] F(x, f(x)) \\
a_{10} + a_{01} c_1 &= [x^1] F(x, f(x)) \\
a_{01} c_2 + a_{20} + a_{11} c_1 + a_{02} c_1^2 &= [x^2] F(x, f(x)) \\
a_{01} c_3 + a_{11} c_2 + 2 a_{02} c_1 c_2 &= [x^3] F(x, f(x)) \\
a_{02} c_2^2 + 2 a_{02} c_1 c_3 + a_{11} c_3 + a_{01} c_4 &= [x^4] F(x, f(x))
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a_{02}c_2^2 + 2a_{02}c_1c_3 + a_{11}c_3 + a_{01}c_4 &= [x^4]F(x, f(x))
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4. In practice, to determine a single $F$, we add a constraint of the form $a_{00} + a_{10} + a_{01} + a_{20} + a_{11} + a_{02} = r$, for some random $r$.

# Approximant of degree 2 (cntd)

In matrix notation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 1 & c_2 & c_1 & c_1^2 \\ 0 & 0 & 0 & c_3 & c_2 & 2c_1c_2 \\ 0 & 0 & 0 & c_4 & c_3 & 2c_1c_3 + c_2^2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{01} \\ a_{11} \\ a_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Approximant of degree 2 (cntd cntd)

$$a_{0,0} = 0$$

$$a_{1,0} = \frac{c_1 c_2^3}{c_1^2 c_2 c_4 - c_1^2 c_3^2 + c_1 c_2^3 - c_1 c_2^2 c_3 + c_2^4 - 2c_1 c_2 c_4 + 2c_1 c_3^2 - c_2^3 + c_2^2 c_3 + c_2 c_4 - c_3^2}$$

$$a_{0,1} = \frac{-c_2^3}{c_1^2 c_2 c_4 - c_1^2 c_3^2 + c_1 c_2^3 - c_1 c_2^2 c_3 + c_2^4 - 2c_1 c_2 c_4 + 2c_1 c_3^2 - c_2^3 + c_2^2 c_3 + c_2 c_4 - c_3^2}$$

$$a_{1,1} = \frac{-2c_1 c_2 c_4 + 2c_1 c_3^2 + c_2^2 c_3}{c_1^2 c_2 c_4 - c_1^2 c_3^2 + c_1 c_2^3 - c_1 c_2^2 c_3 + c_2^4 - 2c_1 c_2 c_4 + 2c_1 c_3^2 - c_2^3 + c_2^2 c_3 + c_2 c_4 - c_3^2}$$

$$a_{2,0} = \frac{c_1^2 c_2 c_4 - c_1^2 c_3^2 - c_1 c_2^2 c_3 + c_2^4}{c_1^2 c_2 c_4 - c_1^2 c_3^2 + c_1 c_2^3 - c_1 c_2^2 c_3 + c_2^4 - 2c_1 c_2 c_4 + 2c_1 c_3^2 - c_2^3 + c_2^2 c_3 + c_2 c_4 - c_3^2}$$

$$a_{0,2} = \frac{c_2 c_4 - c_3^2}{c_1^2 c_2 c_4 - c_1^2 c_3^2 + c_1 c_2^3 - c_1 c_2^2 c_3 + c_2^4 - 2c_1 c_2 c_4 + 2c_1 c_3^2 - c_2^3 + c_2^2 c_3 + c_2 c_4 - c_3^2}$$

# Matrix representation for a cubic approximant

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & c_2 & c_1 & 0 & c_1^2 & 0 & 0 \\
0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & 2c_1c_2 & c_1^2 & c_1^3 \\
0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & 2c_1c_3 + c_2^2 & 2c_1c_2 & 3c_1^2c_2 \\
0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & 2c_1c_4 + 2c_2c_3 & 2c_1c_3 + c_2^2 & 3c_1^2c_3 + 3c_1c_2^2 \\
0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & 2c_1c_5 + 2c_2c_4 + c_3^2 & 2c_1c_4 + 2c_2c_3 & 3c_1^2c_4 + 6c_1c_2c_3 + c_2^3 \\
0 & 0 & 0 & 0 & c_7 & c_6 & c_5 & 2c_1c_6 + 2c_2c_5 + 2c_3c_4 & 2c_1c_5 + 2c_2c_4 + c_3^2 & 3c_1^2c_5 + 6c_1c_2c_4 + 3c_1c_3^2 + 3c_2^2c_3 \\
0 & 0 & 0 & 0 & c_8 & c_7 & c_6 & 2c_1c_7 + 2c_2c_6 + 2c_3c_5 + c_4^2 & 2c_1c_6 + 2c_2c_5 + 2c_3c_4 & 3c_1^2c_6 + 6c_1c_2c_5 + 6c_1c_3c_4 + 3c_2^2c_4 + 3c_2c_3^2
\end{pmatrix}
$$

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# Osculating Spaces

## Definition

The **osculating space** of the analytic curve $X$ of order $k$, degree $d$ at the origin is the $\mathbb{C}$-vector space

$$V_{d,k} = \{F \in \mathbb{C}[x,y] \mid \deg(F) \leq d,\ C(F,X) \geq k\},$$

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## Obvious properties

1. As a vector space, $V_{d,k}$ has at least zero as a member.
2. Recall $F(x, f(x)) = \sum_{j=0}^{\infty} h_j(\mathbf{a},\mathbf{c})x^j$.
3. Hence, we have: $F \in V_{d,k} \implies h_j(\mathbf{a},\mathbf{c}) = 0, 1 \leq j \leq k$.
4. Therefore, we have:

$$V_{d,k+1} \subseteq V_{d,k}.$$

# Osculating curves

## Definition

1. Let $X$ be an analytic curve passant through the origin, let $F(x,y) \in \mathbb{C}[x,y]$ be an **irreducible** polynomial without constant term and let $d$ a positive integer.

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1. Let $X$ be an analytic curve passant through the origin, let $F(x,y) \in \mathbb{C}[x,y]$ be an **irreducible** polynomial without constant term and let $d$ a positive integer.
2. If $V_{d,m} = \mathrm{span}(F(x,y))$ for some $m \in \mathbb{N}$ then we call $F(x,y)$ the degree $d$ *osculating curve* of $X$ at the origin.

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## Remark

As we shall see:

1. an osculating curve of degree $d$ may not exist,
2. but generically, it does exist.

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\begin{aligned}
a_{00} &= 0 \\
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   - 3.3 Increasing $k$ by 1 adds a new equation which involves the parameter $c_k$, which does not appear in $h_j(\mathbf{a}, \mathbf{c})$ for $1 \leq j < k$.
   - 3.4 Hence, increasing $k$ by 1, adds a new constraint which is, generically, linearly independent of the previous ones.

# Algorithm 1: computing osculating spaces

**Input:** The positive integers $d, k$ and the polynomial part of the power series $f(x) = \sum_{i=1}^{\infty} c_i x^i$ of degree $k - 1$.

**Output:** A representation of $theosculatingspaceV_{d,k}$ of $f(x)$.

1. Let $F_d(x, y) = \sum_{1 \leq i+j \leq d} a_{ij} x^i y^j$ with the $a_{ij}$ left as indeterminants
2. Obtain the first $k$ coefficients $h_j(\mathbf{a}, \mathbf{c})$ of $F_d(x, f_{k-1}(x)) = \sum_{j=1}^{\infty} h_j(\mathbf{a}, \mathbf{c}) x^j$.
3. write the matrix $M$ of the system $h_1(\mathbf{a}, \mathbf{c}) = \cdots = h_k(\mathbf{a}, \mathbf{c}) = 0$.
4. Compute and **return** the nullspace of $M$.

# Algorithm 2: computing osculating curves

**Input:** The positive integer $d$ and the power series $f(x) = \sum_{i=1}^{\infty} c_i x^i$

**Output:** If it exists, the osculating curve of degree $d$ for $X$ at the origin.

1. Set $k = N_d$ and compute $V = V_{d,k}$
2. **while** $\dim(V) > 1$ **do**
   - 2.1 Let $k = k + 1$
   - 2.2 Compute $V_{d,k}$
   - 2.3 Let $V = V_{d,k}$
3. **return** a basis $F(x, y)$ for $V$

# The inclusion $V_{d,k+1} \subseteq V_{d,k}$ need not be strict.

1. Consider $y = f(x) = x + x^2 + 2x^3 + 3x^4 + 2x^5$ and $d = 2$.
2. We have $N_d = 5$
3. It turns out that both $V_{2,4}$ and $V_{2,5}$ are equal and that the algorithm needs a second iteration, terminating with $V_{2,6}$ generated by

$$F(x, y) = -x + y + 2x^2 - 4xy + y^2.$$

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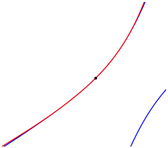
4. Whereas, generically, we expect: $\dim(V_{d,k}) = \dim(V_{d,k+1}) + 1$.
5. For that reason, we say that $V(F)$ a **sextactic contact** with the graph of $f(x)$.

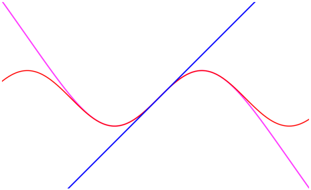


# Osculating curves of degree $d$ may not exist

1. Consider $f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.
2. The point $p = (0, 0)$ on this graph is an inflection point. The tangent line defined by $F(x, y) = x - y$ meets the graph to order 3.
3. It turns out that $V_{2,5} = \text{span}(G(x, y))$ with $G(x, y) = (x - y)^2$.
4. Hence, no osculating conic at the origin.
5. Considering $d = 3$, thus $N_d = 9$, we have:

   $$V_{3,9} = \mathrm{span}((x-y)^3, -42000x+42000y+4437x^3+3159x^2y-729xy^2+133y^3)$$

   and
   $$V_{3,10} = \mathrm{span}(-42000x+42000y+4437x^3+3159x^2y-729xy^2+133y^3).$$



# Plan

# Concluding remarks

## Summary and notes

For a complex analytic curve $X$ defined around a point (the origin) by a power series expansion, and for a degree $d$:

1. We define the osculating curve of degree $d$ for $X$ at that point as the algebraic curve of degree $d$ that has maximum contact order with $X$ at that point.

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2. We demonstrate that, generically, this osculating curve exists.

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3. We propose an algorithm to compute this osculating curve, provided it exists.

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## Work in progress

1. Osculating curves for analytic space curves
2. Osculating surfaces

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