What's new in BPAS?

Basic Polynomial Algebra Subprograms

M. Asadi, **A. Brandt**, R.-J. Jing, M. Kazemi, D. Mohajerani,
R.H.C. Moir, M. Moreno Maza, D. Talaashrafi, L. Wang

Ontario Research Center for Computer Algebra
Department of Computer Science
University of Western Ontario, Canada

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- Fourier-Motzkin elimination: new algorithm, complexity measures [8]

C++ Templates for a Dynamic, Friendly Interface [5]

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$$\text{field} \subset \text{Euclidean domain} \subset \text{GCD domain} \subset \text{integral domain} \subset \text{ring}$$

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→ e.g. $\mathbb{Z}/17\mathbb{Z}$, $\mathbb{Q}[x]$ are both Euclidean domains, but incompatible.

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Solution: *curiously recurring template pattern*, template metaprogramming

→ Abstract class hierarchy encodes the interface of each algebraic type

→ Template parameter makes interface mathematically safe at compile-time

→ Polys automatically decide algebraic type (superclass) based on ground ring

```
template <class Derived>
class BPASRing {    Derived add(Derived x, Derived y)    };

template <class Derived>
class BPASEuclideanDomain : BPASGCDDomain<Derived>;

class Integer : BPASEuclideanDomain<Integer>;

template <class Ring, Deriverd>
class Polynomial : conditional< is_base_of<Ring, BPASField<Ring>,
     BPASEuclideanDomain<Dervied>, BPASRing<Derived> >;
```

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- $\rightarrow$ per term data locality (cf. separate locality for coefficients and monomials)
- $\rightarrow$ optimizes Monagan's heap arithmetic [9]; extends to pseudo-division & n.f.
- $\rightarrow$ *dynamic unpacking* (unpublished) supports *unlimited* variables (cf. 31 in [9])

$$13x^2y^3z^2 + 5x^2y + 7y^3z + 11yz^4 := \underbrace{\boxed{13}\boxed{2|3|2}}_{\text{Term 1}}\underbrace{\boxed{5}\boxed{2|1|0}}_{\text{Term 2}}\underbrace{\boxed{7}\boxed{0|3|1}}_{\text{Term 3}}\underbrace{\boxed{11}\boxed{0|1|4}}_{\text{Term 4}}$$

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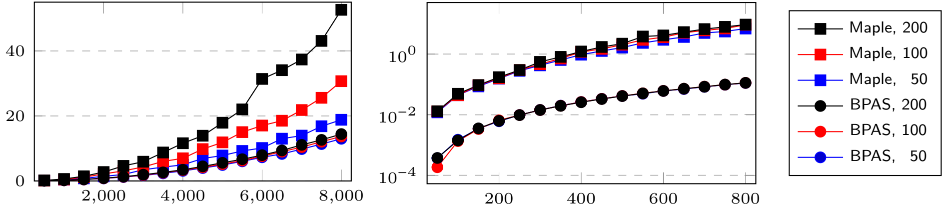
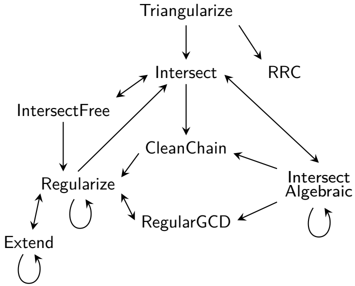
$\mathbb{Z}[X_1, \ldots, X_5]$ multiplication and Euclidean division (time (s) vs number of terms; varying sparsity)



Parallel Triangular Decompositions [2]

Goal: parallelism in triangular decomposition despite geometric problems



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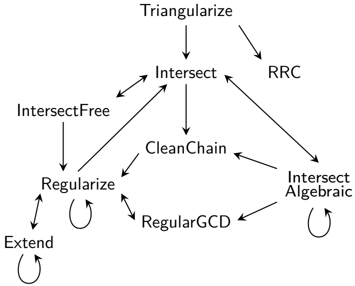
→ Fine-grained: *asynchronous generators* form a pipeline among subroutines.

→ DnC: removal of redundant components



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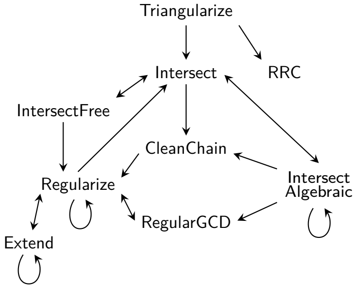

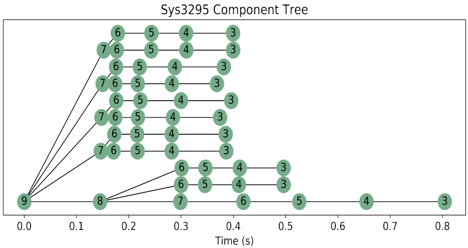

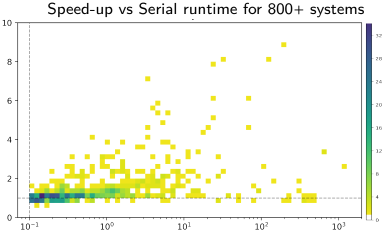

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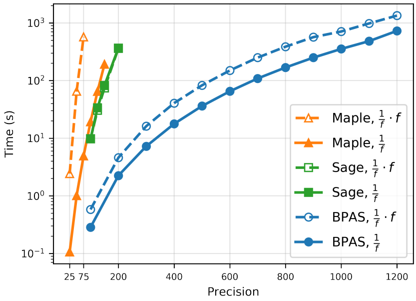
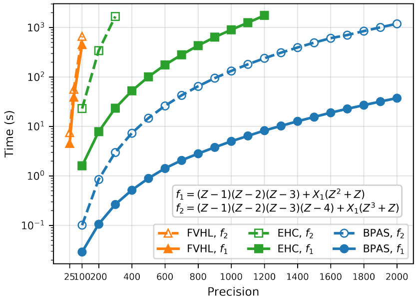
Power Series Arithmetic for $f = 1 + X_1 + X_2 + X_3$



Factorization via Hensel's Lemma in $\mathbb{Q}[[X_1]][Z]$



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[2] M. Asadi, A. Brandt, R. H. C. Moir, M. Moreno Maza, and Y. X. "On the Parallelization of Triangular Decomposition of Polynomial Systems". In: *ISSAC 2020, Proceedings*. (to appear). 2020.

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