The Delinearization of *C* programs

Chirantan Mukherjee, Marc Moreno Maza, Linxiao Wang

Ontario Research Centre for Computer Algebra

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Content

One-dimensional array

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1. Can we parallelize the two `for-loops`?
2. Is there data dependence between two different iterations of the nest?
3. Are there integer solutions to the following system of linear inequalities?

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 \times n + j_1 = n \times j_2 - n + j_2 - i_2 - 1 \end{cases}$$

Delinearize the array accesses

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There is no integer solution, therefore, no dependence.

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such that:

$$\boxed{R = f_1 M_2 \cdots M_e \; + \; \cdots \; + \; f_{e-1} M_2 + f_e}$$

for each $(i_1, \ldots, i_d)$ in the iteration domain we have the validity conditions:

$$\boxed{0 \leq f_1 < M_1, \quad \ldots, 0 \leq f_e < M_e.}$$

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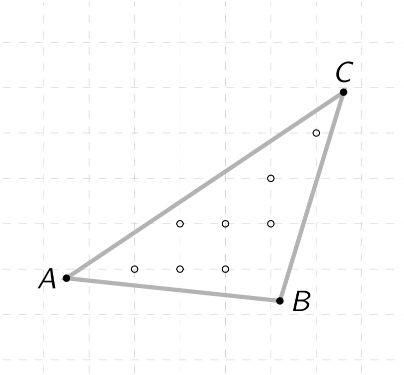

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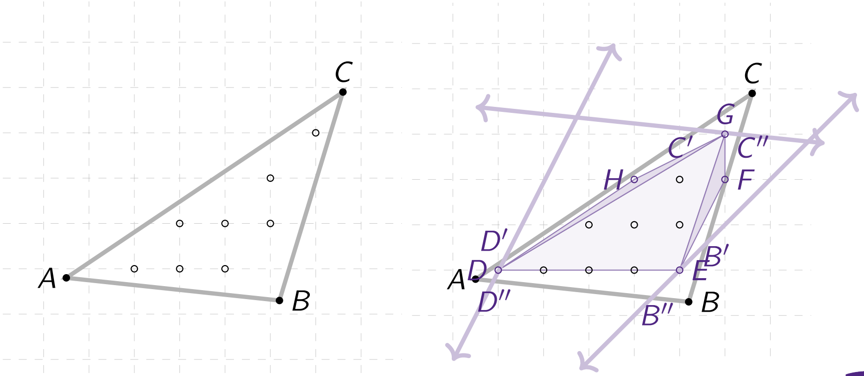

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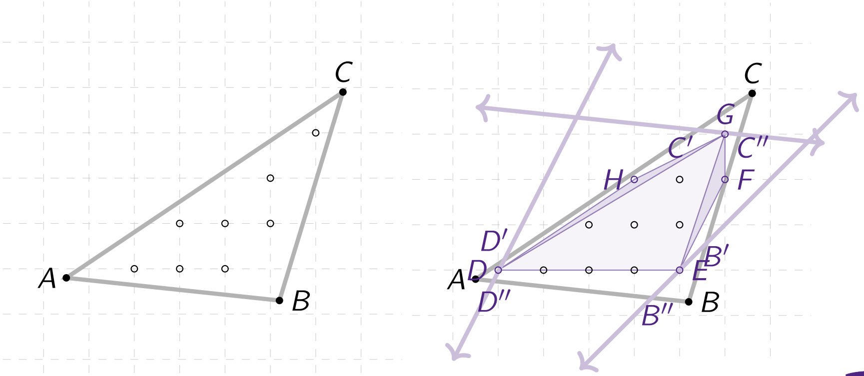

The integer hull $\{D, E, F, G, H\}$ is formed by $\{D, E, G\}$ and searching integer points $\{F, H\}$ in quadrilaterals $DD''B''E$, $EB'C''G$ and $D'DGC'$.

Parametric Integer Linear Programming

$$\begin{array}{ll} \max\limits_{(i_1,\ldots,i_d)} & f_k \\ \text{subject to} & (i_1, \ldots, i_d) \in \text{iteration domain} \\ & i_1, \ldots, i_d \in \mathbb{Z} \end{array}$$

For which, $0 \leq \max\limits_{(i_1,\ldots,i_d)} f_k < M_k$ for all $k \in [1, \ldots, e]$.

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1. `PIP/PipLib` by Paul Feautrier
2. `isl` by Sven Verdoolaege
3. `barvinok` by Sven Verdoolaege

2D-2D delinearization problem

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The reference $A[R]$ to A which encodes a reference $B[f1][f2]$ to B, where:

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$$\left.\begin{array}{lll} d = 2 \rightsquigarrow i_1, i_2 & \text{loop counters} \\ e = 2 \rightsquigarrow m_1, m_2 & \text{program parameters} \end{array}\right\} \text{known at compile time.}$$

$$\left.\begin{array}{l} M_1 = a_1 m_1 + b_1 \\ M_2 = a_2 m_2 + b_2 \end{array}\right\}, \text{ where } a_1, b_1, a_2, b_2 \in \mathbb{Z} \text{ TBD.}$$

The reference $A[R]$ to A which encodes a reference $B[f1][f2]$ to B, where:

- $$\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases} \quad \text{and} \quad R = f1 M_2 + f2$$

2D-2D delinearization problem

Input

```
for(i_1, ..., i_1 ++)
 ...
  for(i_d, ..., i_d ++)
   A[R(i_1, ..., i_d, m_1, ..., m_e)] <- ...
```

Output

```
for(i_1, ..., i_1 ++)
 ...
  for(i_d, ..., i_d ++)
   B[f_1], ..., B[f_e] <- ...
```

$$\left.\begin{array}{lll} d = 2 \rightsquigarrow i_1, i_2 & \text{loop counters} \\ e = 2 \rightsquigarrow m_1, m_2 & \text{program parameters} \end{array}\right\} \text{known at compile time.}$$

$$\left.\begin{array}{l} M_1 = a_1 m_1 + b_1 \\ M_2 = a_2 m_2 + b_2 \end{array}\right\}, \text{ where } a_1, b_1, a_2, b_2 \in \mathbb{Z} \text{ TBD.}$$

The reference $A[R]$ to A which encodes a reference $B[f1][f2]$ to B, where:

- $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ and $R = f1 M_2 + f2$
- for each (i_1, i_2), we have, $\begin{cases} 0 \leq f1 < M_1 \\ 0 \leq f2 < M_2 \end{cases}$, thus $\begin{cases} 0 \leq \max f1 < M_1 \\ 0 \leq \max f2 < M_2 \end{cases}$.

2D-2D polynomial system solving

Substituting $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ in $R = f1 M_2 + f2$, we obtain,

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$$R = \underbrace{a_2 f_{11}}_{T_1} i_1 m_2 + \underbrace{a_2 f_{12}}_{T_2} i_2 m_2 + \underbrace{a_2 f_{10}}_{T_3} m_2 + (\underbrace{b_2 f_{11} + f_{21}}_{T_4}) i_1 + (\underbrace{b_2 f_{12} + f_{22}}_{T_5}) i_2 + (\underbrace{b_2 f_{10} + f_{20}}_{T_6}).$$

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Substituting $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ in $R = f1 M_2 + f2$, we obtain,

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$$\begin{cases} T_1 = a_2 f_{11} \\ T_2 = a_2 f_{12} \\ T_3 = a_2 f_{10} \\ T_4 = b_2 f_{11} + f_{21} \\ T_5 = b_2 f_{12} + f_{22} \\ T_6 = b_2 f_{10} + f_{20} \end{cases} \implies \begin{cases} f_{11} = \dfrac{T_1}{a_2} \\ f_{12} = \dfrac{T_2}{a_2} \\ f_{10} = \dfrac{T_3}{a_2} \\ f_{21} = T_4 - b_2 f_{11} \\ f_{22} = T_5 - b_2 f_{12} \\ f_{20} = T_6 - b_2 f_{10} \end{cases}$$

2D-2D polynomial system solving

Substituting $\begin{cases} f1 = f_{11} i_1 + f_{12} i_2 + f_{10} \\ f2 = f_{21} i_1 + f_{22} i_2 + f_{20} \end{cases}$ in $R = f1 M_2 + f2$, we obtain,

$$R = \underbrace{a_2 f_{11}}_{T_1} i_1 m_2 + \underbrace{a_2 f_{12}}_{T_2} i_2 m_2 + \underbrace{a_2 f_{10}}_{T_3} m_2 + (\underbrace{b_2 f_{11} + f_{21}}_{T_4}) i_1 + (\underbrace{b_2 f_{12} + f_{22}}_{T_5}) i_2 + (\underbrace{b_2 f_{10} + f_{20}}_{T_6}).$$

$$\begin{cases} T_1 = a_2 f_{11} \\ T_2 = a_2 f_{12} \\ T_3 = a_2 f_{10} \\ T_4 = b_2 f_{11} + f_{21} \\ T_5 = b_2 f_{12} + f_{22} \\ T_6 = b_2 f_{10} + f_{20} \end{cases} \implies \begin{cases} f_{11} = \frac{T_1}{a_2} \\ f_{12} = \frac{T_2}{a_2} \\ f_{10} = \frac{T_3}{a_2} \\ f_{21} = T_4 - b_2 f_{11} \\ f_{22} = T_5 - b_2 f_{12} \\ f_{20} = T_6 - b_2 f_{10} \end{cases}$$

a_2, b_2 can NOT be uniquely determined, but $a_2 \mid gcd(T_1, T_2, T_3)$.

2D-2D quantifier elimination (I/II)

For each (i_1, i_2), we have, $0 \leq f2 < M_2$, thus $0 \leq \max f2 < M_2$,

$$\begin{cases} \max\limits_{(i_1,i_2)} & f_2 \\ \text{subject to} & (i_1, i_2) \in \text{iteration domain} \\ & i_1, i_2 \in \mathbb{Z} \end{cases}$$

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Depending on the shape of the iteration domain, we solve on a case to case basis.

2D-2D quantifier elimination (I/II) - Rectangular domain

For a `for loop` of the form,

```
for (int i_1 = 0; i_1 < r_1; i_1 ++)
  for (int i_2 = 0; i_2 < r_2; i_2 ++)
```

2D-2D quantifier elimination (I/II) - Rectangular domain

For a `for loop` of the form,

```
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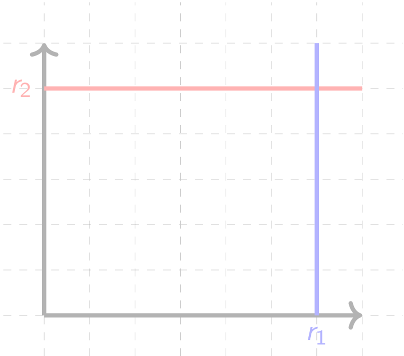
```

the iteration domain is of the shape,



2D-2D quantifier elimination (I/II) - Triangular domain

For a `for loop` of the form,

```
for (int i_1 = 0; i_1 < r_1; i_1 ++)
  for (int i_2 = 0; p * i_1 + q * i_2 < r_2; i_2 ++)
```

2D-2D quantifier elimination (I/II) - Triangular domain

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for (int i_1 = 0; i_1 < r_1; i_1 ++)
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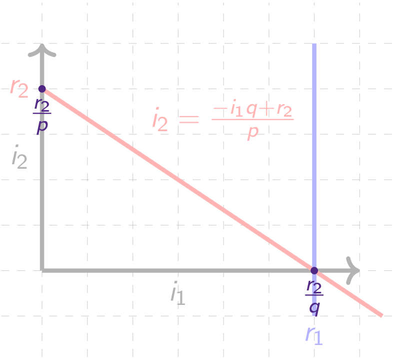

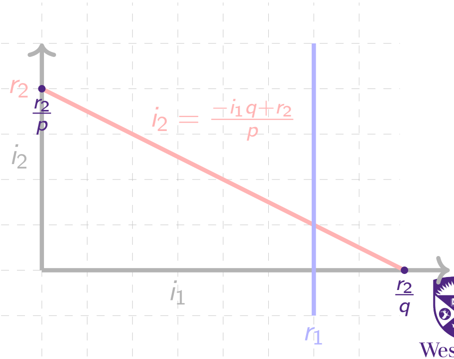

2D-2D quantifier elimination (I/II)

The parametric integer linear problem:

$$\begin{cases} \max\limits_{(i_1,i_2)} & f_2 = f_{21} i_1 + f_{22} i_2 + f_{20} \\ \text{subject to} & (i_1, i_2) \in \text{iteration domain} \\ & i_1, i_2 \in \mathbb{Z} \end{cases}$$

can be solved for,

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can be solved for,

1. Rectangular domain by case inspection

2D-2D quantifier elimination (I/II)

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2. Triangular domain by case inspection except when `f_21, f_22 > 0`, in which case the problem becomes,

2D-2D quantifier elimination (I/II)

The parametric integer linear problem:

$$\begin{cases} \max\limits_{(i_1,i_2)} & f_2 = f_{21} i_1 + f_{22} i_2 + f_{20} \\ \text{subject to} & (i_1, i_2) \in \text{iteration domain} \\ & i_1, i_2 \in \mathbb{Z} \end{cases}$$

can be solved for,

1. Rectangular domain by case inspection
2. Triangular domain by case inspection except when `f_21, f_22 > 0`, in which case the problem becomes,

$$\begin{array}{lll} \max\limits_{i_1} & f_{21} i_1 + f_{22} \lfloor \frac{-i_1 q + r_2}{p} \rfloor + f_{20} & \\ \text{subject to} & 0 \leq i_1 < r_1, & i_1 \in \mathbb{Z} \end{array}$$

2D-2D quantifier elimination (II/II)

QE over $\mathbb{R}$,

2D-2D quantifier elimination (II/II)

QE over $\mathbb{R}$, uses the `QuantifierElimination` function from the `RegularChains:-SemiAlgebraicSetTools` package.

2D-2D quantifier elimination (II/II)

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```
f := &A([i_1, i_2]), ((0 < i_1) &and (i_1 < r_1) &and
                      (0 < i_2) &and (i_2 < r_2) &and
                      (0 < r_1) &and (0 < r_2) &and
                      (0 < f_{21}) &and (0 < f_{22}) &and (0 < B))
                      \\ B = M_2 - f_{20}
            &implies (f_{21} * i_1 + f_{22} * i_2 < B);
```

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                      (0 < f_{21}) &and (0 < f_{22}) &and (0 < B))
                      \\ B = M_2 - f_{20}
             &implies (f_{21} * i_1 + f_{22} * i_2 < B);
```

After simplification, `r_1 * f_{21} + r_2 * f_{22} + f_{20} < M_2`.

2D-2D quantifier elimination (II/II)

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                      (0 < i_2) &and (i_2 < r_2) &and
                      (0 < r_1) &and (0 < r_2) &and
                      (0 < f_{21}) &and (0 < f_{22}) &and (0 < B))
                      \\ B = M_2 - f_{20}
           &implies (f_{21} * i_1 + f_{22} * i_2 < B);
```

After simplification, `r_1 * f_{21} + r_2 * f_{22} + f_{20} < M_2`. Substituting, $f_{11} = \frac{T_1}{a_2}, f_{12} = \frac{T_2}{a_2}, f_{10} = \frac{T_3}{a_2}, f_{21} = T_4 - b_2 f_{11}, f_{22} = T_5 - b_2 f_{12}, f_{20} = T_6 - b_2 f_{10}, M_2 = a_2 m_2 + b_2$, we obtain,

$$\boxed{r_1(T_4 - b_2\frac{T_1}{a_2}) + r_2(T_5 - b_2\frac{T_2}{a_2}) + T_6 - b_2\frac{T_3}{a_2} < a_2 m_2 + b_2.}$$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = r_2$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
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    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = r_2$
2. $T_1 = 2$, $T_2 = 0$, $T_3 = 1$, $T_4 = 0$, $T_5 = 3$, $T_6 = 2$

Examples (I/II) - Rectangular domain

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for (int i_1 = 0; i_1 <= r_1; i_1 ++)
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```

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3. $a_2 =?, b_2 =?, f_{11} = 2, f_{12} = 0, f_{10} = 1, f_{21} = -2b_2, f_{22} = 3, f_{20} = 2 - b_2$

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4. the validity condition $-r_1 b_2 \frac{2}{a_2} + 3r_2 + 2 - b_2 \frac{1}{a_2} < a_2 m_2 + b_2$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
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5. evaluating at $a_2 = 1, b_2 = 0$, we obtain, $f1 = 2i_1 + 1, f2 = 3i_2 + 2$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

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5. evaluating at $a_2 = 1, b_2 = 0$, we obtain, $f1 = 2i_1 + 1, f2 = 3i_2 + 2$
6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,
   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 5 < 10$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = r_2$
2. $T_1 = 2, T_2 = 0, T_3 = 1, T_4 = 0, T_5 = 3, T_6 = 2$
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6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,

   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 5 < 10$

   delinearization valid when $r_1 = r_2 = 2, max\ f2 = 8 < 10$

Examples (I/II) - Rectangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = r_2$
2. $T_1 = 2, T_2 = 0, T_3 = 1, T_4 = 0, T_5 = 3, T_6 = 2$
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5. evaluating at $a_2 = 1, b_2 = 0$, we obtain, $f1 = 2i_1 + 1, f2 = 3i_2 + 2$
6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,

   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 5 < 10$
   delinearization valid when $r_1 = r_2 = 2, max\ f2 = 8 < 10$
   delinearization invalid when $r_1 = r_2 = 3, max\ f2 = 11 \not< 10$

Examples (II/II) - Triangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
  for (int i_2 = 0; i_1 + 2 * i_2 <= r_2; i_2 ++)
    A[2 * i_1 * m_2 + m_2 + 3 * i_2 + 2] = ...;
```

1. $max\ i_1 = r_1, max\ i_2 = \lfloor \frac{r_2}{2} \rfloor$

Examples (II/II) - Triangular domain

```
for (int i_1 = 0; i_1 <= r_1; i_1 ++)
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Examples (II/II) - Triangular domain

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Examples (II/II) - Triangular domain

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1. $max\ i_1 = r_1, max\ i_2 = \lfloor \frac{r_2}{2} \rfloor$
2. $T_1 = 2$, $T_2 = 0$, $T_3 = 1$, $T_4 = 0$, $T_5 = 3$, $T_6 = 2$
3. $a_2 =?, b_2 =?, f_{11} = 2, f_{12} = 0, f_{10} = 1, f_{21} = -2b_2, f_{22} = 3, f_{20} = 2 - b_2$
4. the validity condition $-r_1 b_2 \frac{2}{a_2} + 3r_2 + 2 - b_2 \frac{1}{a_2} < a_2 m_2 + b_2$
5. evaluating at $a_2 = 1, b_2 = 0$, we obtain, $f1 = 2i_1 + 1, f2 = 3i_2 + 2$
6. assuming $m_2 = 10$, i.e. $B[...][10]$, we get,
   delinearization valid when $r_1 = r_2 = 1, max\ f2 = 2 < 10$

Examples (II/II) - Triangular domain

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Examples (II/II) - Triangular domain

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   delinearization valid when $r_1 = r_2 = 4, max\ f2 = 8 < 10$
   delinearization valid when $r_1 = r_2 = 5, max\ f2 = 8 < 10$
   delinearization invalid when $r_1 = r_2 = 6, max\ f2 = 11 \not< 10$

Concluding remarks

Summary and notes

1. In the area of optimizing compilers, delinearization of 1D array is a necessary step before applying the techniques of the polyhedral model.
2. Current compilers use heuristics fail to delinearize some for-loop nests.
3. Solving the delinearization problem is an algebraic problem with two sub-problems: a *polynomial system solving* one and a *QE* one.
4. It is desirable to solve them *at compile time* as much as possible, although some parameters are only known at *execution time*.
5. We have shown that this is indeed possible for some classes of delinearization problems (2D-2D and specific iterations domains)

Work in progress

1. We are currently extending our results to higher dimension
2. Solving the QE problem at compile time requires to improve existing techniques for PILP (parametric integer linear programming) and/or Presburger arithmetic.

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