DIFFERENTIABLE MANIFOLDS AND FORMS, DE RHAM COHOMOLOGY

– Project –

for

Curves and Surfaces with Mathematica

submitted by

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Preface

This note is part of my project on "Differentiable Manifolds and Forms, de Rham Cohomology" that I delivered on the January, 15 2021.

I have tried to provide a very "generalized" theory of the real manifolds (but almost all the theory follows in the complex case with necessary changes) and covered almost all background knowledge that would be required to understand about differential forms and de Rham cohomology without any difficulty. There are 3 propositions in this notes, for which I have intentionally skipped the proof, because understanding the proof requires sophisticated techniques which we are not going to explore in this note. I have also tried to make this note error-free to the best of my ability. But if you notice any error or any inconsistency, please let me know.

Lastly, thank you for choosing to read my note. I hope you will enjoy it :)

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Contents

1 Manifold

1.1 Topological Manifold

We first introduce the model space in the real case.

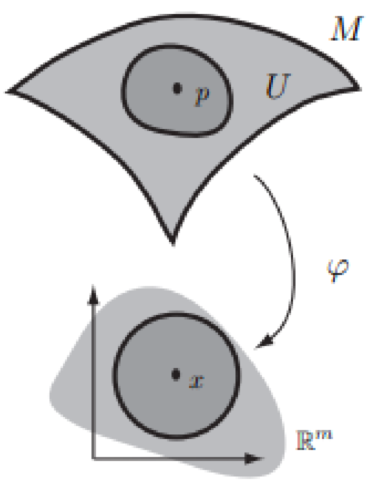
Definition 1.1. *A* **topological manifold with boundary** *of dimension n is a topological space $\mathcal{M}$ which:*

1. *is locally homeomorphic to $\mathbb{R}^n_\pm$ i.e. $\forall p \in \mathcal{M}$, $\exists U(p)$ open and $p \in U(p)$ such that $U(p) \cong V \subset_{open} \mathbb{R}^n_\pm$*
2. *is Hausdorff*
3. *is connected*
4. *is second countable*

NOTE: The symbol $\mathbb{R}^n_\pm$ means: $\mathbb{R}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\}$, $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, $\mathbb{R}^n_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$.

Example 1. *Every open set of $\mathbb{R}^n_\pm$ is a topological manifold with boundary. We can choose an atlas which consists of only one coordinate chart $U = \mathbb{R}^n_\pm$, and $\varphi = Id$.*

Definition 1.2. *A* **chart** *(U, φ) on $\mathcal{M}$ is given by an open set $U \subset \mathcal{M}$ and an homeomorphism $\varphi \colon U \to D$, onto an open set D of $\mathbb{R}^m_\pm$.*



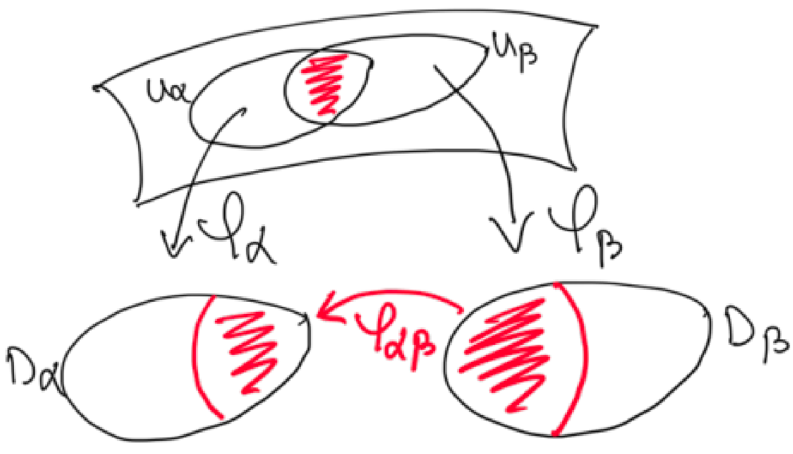
While using a paper map, we have to move from one map to another. To follow our path we need to find the coordinates, in both maps, of the same point of our position in that moment. The way to do so, is the following:

Definition 1.3. *For every ordered pair of charts* $(U_\alpha, \varphi_\alpha)$ *and* (U_β, φ_β)*, the set of* $\{\varphi_{\alpha\beta}\}$ *is the set of* **transition functions**, $\varphi_{\alpha\beta} \coloneqq \varphi_\alpha \circ \varphi_\beta^{-1}$,
i.e. $\varphi_{\alpha\beta} = \varphi_\alpha \mid_{U\alpha \cap U_\beta} \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)}$
$\varphi_{\alpha\beta} \colon \varphi_\beta(U\alpha \cap U_\beta) \to \varphi_\alpha(U\alpha \cap U_\beta)$.

Definition 1.4. *An* **atlas** *for a topological space* $\mathcal{M}$ *is a family of charts* $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathcal{I}}$ *on* $\mathcal{M}$ *such that* $\bigcup_{\alpha \in \mathcal{I}} U_\alpha = \mathcal{M}$ *and all transition functions* $\varphi_{\alpha\beta} \coloneqq \varphi_\alpha \circ \varphi_\beta^{-1}$ *are smooth.*

The transition function satisfies the following **cocylce conditions**:

1. $\forall \alpha \in \mathcal{I}$, $\varphi_{\alpha\alpha} = Id$
2. $\forall \alpha, \beta \in \mathcal{I}$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$
3. $\forall \alpha, \beta, \gamma \in \mathcal{I}$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$



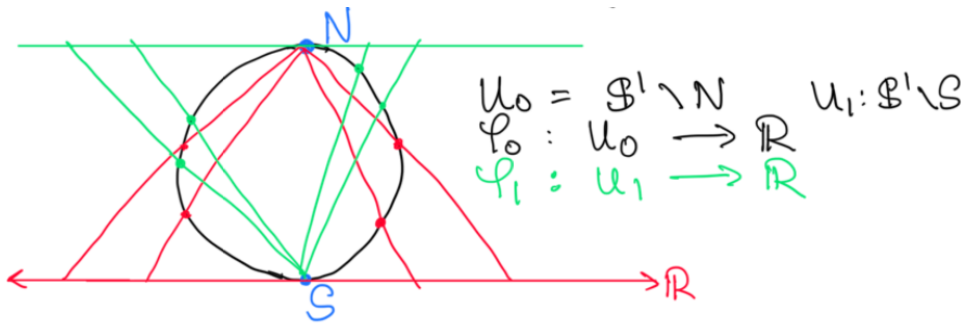
Example 2. *Let* $\mathcal{M} = \mathbb{R}^n$ *and take* $U = \mathcal{M}$ *with* $\varphi = Id$*. We could also take* $\mathcal{M}$ *to be any open set in* $\mathbb{R}^n$*.*

A chart allows to use the coordinates of $\mathbb{R}^n$ to identify a point of the mapped object U of the manifold $\mathcal{M}$.
From now on we will denote by u_i the i-th coordinate function on $\mathbb{R}^n$

$u_i \colon \mathbb{R}^n \to \mathbb{R}$
$(w_1, \dots, w_n) \to w_i$.

Each chart (U, φ) induces **local coordinates** $(x_1, \dots, x_n)$ defined by
$x_i \coloneqq u_i \circ \varphi \colon U \to \mathbb{R}$.

Example 3. $\mathbb{S}^1 = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\}$
$U_0 = \mathbb{S}^1 \setminus N$ *and* $\varphi_0 \colon U_0 \to \mathbb{R}$
$U_1 = \mathbb{S}^1 \setminus N$ *and* $\varphi_1 \colon U_1 \to \mathbb{R}$
transition function φ_{01} *is* $\mathcal{C}^\infty$.



Example 4. $\mathbb{S}^n = \{(x_0, \dots, x_n) \mid \Sigma_{i=0}^{i=n} x_i = 1\}$
where north pole $N = (1, \dots, 0)$ *and south pole* $S = (-1, \dots, 0)$
charts $U_0 = \mathbb{S}^n \setminus N$ *and* $U_1 = \mathbb{S}^n \setminus S$
and stereographic projections $\varphi_0 \colon U_0 \to \mathbb{R}^n$ *is an isomorphism and* $\varphi_1 \colon U_1 \to \mathbb{R}^n$ *is an isomorphism.*

Definition 1.5. *A chart* (V, ψ) *is* **compatible** *with an atlas* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}}$ *if* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}} \bigcup \{V, \psi\}$ *is still an atlas.*

Similarly, two atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}}$ *and* $\{(V_\beta, \psi_\beta)\}_{\beta \in \mathcal{I}'}$ *are said to be* **compatible** *if* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}} \bigcup \{(V_\beta, \psi_\beta))\}_{\beta \in \mathcal{I}'}$ *is still an atlas.*

Two atlases are said to be **equivalent** if their union is also an atlas.

1.2 Differentiable Manifold

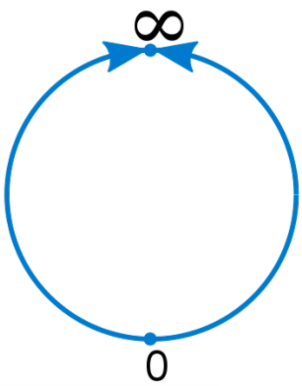
Definition 1.6. *A* **differentiable structure** *on* $\mathcal{M}$ *is an equivalence class of atlases.*

Definition 1.7. *An* **differentiable manifold** *is a topological manifold* $\mathcal{M}$ *with a differentiable structure.*

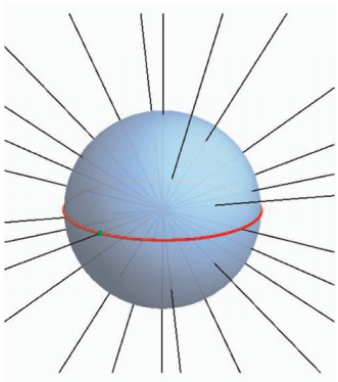
Example 5. *All the examples in the last section are also examples of an differentiable manifold.*

Example 6. *The* **n-dimensional real projective space** $\mathbb{P}^{\mathbf{n}}_{\mathbb{R}} := \mathbb{P}(\mathbb{R}^{\mathbf{n+1}})$.
We say that a point $p \in \mathbb{P}^n_{\mathbb{R}}$ *has homogeneous coordinates* $(x_0\colon \ldots : x_n)$ *if* p *is the class of points* $(x_0, \ldots, x_n) \subset \mathbb{R}^{n+1}$.
Note that every point can be represented by infinitely many different coordinates pairwise related by the multiplication of a scalar. However,

1. *The open set* $U_j := \{x_j \neq 0\}$ *is well defined* $\forall j$.
2. *The maps* $\varphi_j \colon U_j \to \mathbb{R}^n$ *defined by,*
   $\varphi_j(x_0\colon \ldots : x_n) = (\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j})$
   are well defined.
3. (U_j, φ_j) *is an atlas for* $\mathbb{P}^n_{\mathbb{R}}$.



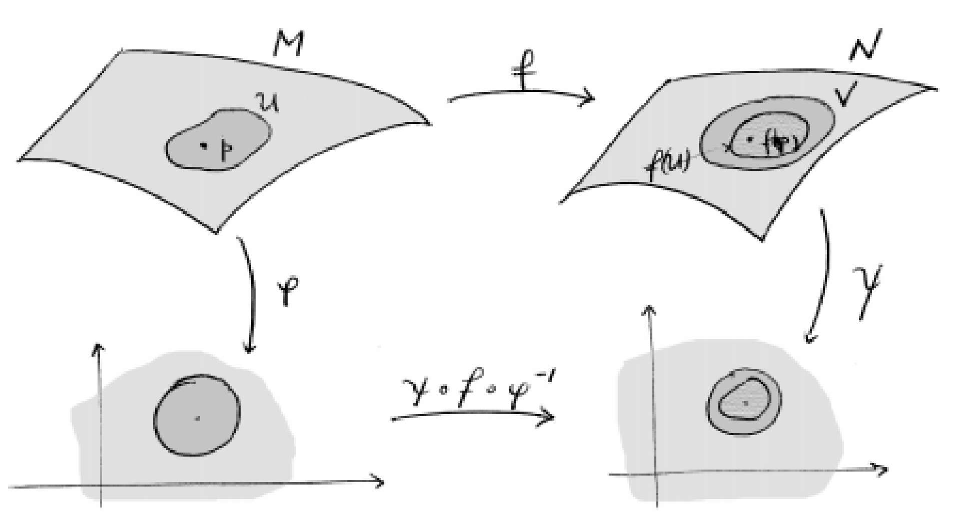
$\mathbb{P}^{\mathbf{1}}_{\mathbb{R}}$


$\mathbb{P}^{\mathbf{2}}_{\mathbb{R}}$

Definition 1.8. *Let* U *be an open set of* $\mathbb{R}^n_{\pm}$. *A function* $F\colon U \to \mathbb{R}^m$ *is* **smooth** *if there is an open set* $V \subset \mathbb{R}^n$ *with* $V \cap \mathbb{R}^n_{\pm} = U$ *and a smooth function* $G\colon V \to \mathbb{R}^m$ *which extends* F, *i.e. such that* $G\,|_U = F$.

Definition 1.9. *Let* $\mathcal{M}$ *be a manifold with atlas* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{I}}$ *and* $\mathcal{N}$ *a manifold with atlas* $\{(V_\beta, \psi_\beta)\}_{\beta\in\mathcal{I}'}$.
A function $f\colon \mathcal{M} \to \mathcal{N}$ *is* **smooth in a point p** $\in \mathcal{M}$ *if, given a chart* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{I}}$ *with* $p \in U_\alpha$, *and a chart* $\{(V_\beta, \psi_\beta)\}_{\beta\in\mathcal{I}'}$ *with* $f(p) \in V_\beta$,*the function* $\psi_\beta \circ f \circ \varphi_\beta^{-1}$ *is smooth.*

Example 7. *An important example of* $\mathcal{C}^\infty$ *functions is a* **bump function** *on a manifold* $\mathcal{M}$. *More precisely, for any open sets* $U, V \subset \mathcal{M}$ *with* U *compact and*



$U \subset V$, there exists some $f \in C^\infty(\mathcal{M})$, such that

$$f(x) = \begin{cases} 1 & x \in \overline{U}, \\ 0 & q \notin V. \end{cases}$$

Example 8. *The natural projection $\pi_1 \colon \mathcal{M} \times \mathcal{N} \to \mathcal{M}$ and $\pi_2 \colon \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ given by $\pi_1(x, y) \mapsto x$ and $\pi_2 \colon (x, y) \mapsto y$ are smooth maps.*

Example 9. *Another important example of $\mathcal{C}^\infty$ maps is a* **smooth curve** *on a manifold $\mathcal{M}$ a $\mathcal{C}^\infty$ map from some open interval $I \subset \mathbb{R}$ to $\mathcal{M}$.*

Definition 1.10. *A* **diffeomorphism** *is a smooth function which is invertible and whose inverse function is also smooth.*

Example 10. *Consider $\mathcal{N} = \mathbb{R}$ with atlas $(\mathbb{R}, \varphi(p) = p)$, and $\mathcal{M} = \mathbb{R}$ with atlas $(\mathbb{R}, \psi(q) = q^3)$. Clearly these define different differentiable structures (non-compatible charts). Between $\mathcal{N}$ and $\mathcal{M}$ we consider the mapping $f(p) = p^{\frac{1}{3}}$, which is a homeomorphism between $\mathcal{N}$ and $\mathcal{M}$. The claim is that f is also a diffeomorphism. Take $U = V = \mathbb{R}$, then $\psi \circ f \circ \varphi^{-1}(p) = ((p^{\frac{1}{3}})^3) = p$ is the identity and thus $\mathcal{C}^\infty$ on $\mathbb{R}$, and the same for $\varphi \circ f^{-1} \circ \psi^{-1}(q) = ((q^3)^{\frac{1}{3}}) = q$. The associated differentiable structures are diffeomorphic. In fact the above described differentiable structures correspond to defining the differential quotient via $lim_{h \to 0} \dfrac{f^3(p+h) - f^3(p)}{h}$.*

2 Forms

2.1 Tensor Algebra

In this section we develop some tools in advanced linear algebra.

Let $V_1, \ldots, V_q$ be finite dimensional vector spaces over a field $\mathbb{K}$ of characteristic 0; this includes $\mathbb{R}$ and $\mathbb{C}$.

Definition 2.1. *A map $\omega\colon V_1 \times \ldots \times V_q \to \mathbb{K}$. is* **multilinear** *or* **q-linear** *or* **tensor of degree q** *if the following holds: $\forall i \in \{1, \ldots, q\}$ and $\forall j \neq i$ of vectors $v_j \in V_j$ the induced map $\psi\colon V_i \to \mathbb{K}$ defined by $\forall v_i \in V_i$, $\psi(v) = (v_1 \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_q)$ is linear.*

Example 11. *The tensors of degree 1 form the dual space V_1^* of V_1.*

Example 12. *The tensors of degree 2 are bilinear maps as:*
$\omega\colon V \times W \to \mathbb{R}$
$\omega(\lambda v_1 + \mu v_2, w) = \lambda\omega(v_1, w) + \mu\omega(v_2, w)$
$\omega(v, \lambda w_1 + \mu w_2) = \lambda\omega(v, w_1) + \mu\omega(v, w_2)$
λ and μ are scalars in $\mathbb{K}$.

Example 13. *For every $n \geq 1$ the map $det\colon (\mathbb{R}^n)^n \to \mathbb{R}$ associating, to each ordered list of n vectors in $\mathbb{R}^n$, the determinant of the matrix whose columns are them, in the same order is a tensor of degree n.*

Definition 2.2. *The space of multilinear maps from $V_1 \times V_2 \times \ldots V_q$ to $\mathbb{K}$ is a vector space, which is the* **tensor product** *of $V_1^*, V_2^*, \ldots, V_q^*$ and is denoted by $V_1^* \otimes V_2^* \otimes \ldots \otimes V_q^*$.*

Definition 2.3. *Choose $\forall i, 1 \leq i \leq q$, an element $\varphi_i \in V_i^*$*
Then define $\varphi_1 \otimes \ldots \otimes \varphi_q$ by
$\varphi_1 \otimes \ldots \otimes \varphi_q(v_1, \ldots, v_q) = \varphi_1(v_1) \cdot \varphi_2(v_2) \cdot \ldots \cdot \varphi_q(v_q)$.
These are the **decomposable tensors** *in $V_1^* \otimes V_2^* \otimes \ldots \otimes V_q^*$.*

NOTE: If $V_1 = V_2 = \ldots = V_q =: V$ then we write $(V^*)^{\otimes q} := V_1^* \otimes V_2^* \otimes \ldots \otimes V_q^*$ for short.

Definition 2.4. *A tensor $\omega \in (V^*)^{\otimes q}$*

1. *is* **symmetric** *if $\forall i \neq j$,*
   $\omega(\ldots, v_i, \ldots, v_j, \ldots) = \omega(\ldots, v_j, \ldots, v_i, \ldots)$
   The symmetric tensors form a vector subspace of $(V^*)^{\otimes q}$ usually denoted $Sym^q V^*$

2. *is* **alternating** *or* **skew** *if* $\forall i \neq j$,
   $\omega(\ldots, v_i, \ldots, v_j, \ldots) = -\omega(\ldots, v_j, \ldots, v_i, \ldots)$
   The skew tensors form a vector subspace of $(V^*)^{\otimes q}$ *usually denoted* $\Lambda^q V^*$

NOTE: We will concentrate only on skew tensors $\Lambda^q V^*$ in this project!

1. $Sym^q V^* \bigcap \Lambda^q V^* = \{0\}, \forall q \geq 2$

2. $Sym^0 V^* = \mathbb{K} = \Lambda^0 V^*$

3. $Sym^1 V^* = V^* = \Lambda^1 V^*$

Definition 2.5. $\forall \varphi_1, \varphi_2 \in V^*$, *and* $v_1, v_2 \in V$ *we define the* **wedge product**, $\varphi_1 \wedge \varphi_2 = \frac{1}{2}(\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1) \in \Lambda^2 V^*$ *as,*
$\wedge \colon \Lambda^1 V^* \times \Lambda^1 V^* \to \Lambda^2 V^*$
$(\varphi_1, \varphi_2) \mapsto \varphi \wedge \varphi_2$
where,
$\varphi \wedge \varphi_2 \colon V \times V \to \mathbb{K}$
$(v_1, v_2) \mapsto \frac{1}{2}\, det \begin{pmatrix} \varphi_1(v_1) & \varphi_1(v_2) \\ \varphi_2(v_1) & \varphi_2(v_2) \end{pmatrix}$

There is a natural extension of this idea to the $\Lambda^q V^*$.

Definition 2.6. *We define the* **wedge product** *as*
$\wedge \colon \Lambda^{q_1} V^* \times \Lambda^{q_2} V^* \to \Lambda^{q_1+q_2} V^*$
$(\omega_1, \omega_2) \mapsto \omega \wedge \omega_2$
where,
$\omega_1 \wedge \omega_2 \colon V^{q_1} \times V^{q_2} \to \mathbb{K}$
$$(v_1, \ldots, v_{q_1+q_2}) \mapsto \frac{1}{(q_1+q_2)!} \sum_{\sigma \in \Sigma_{q_1+q_2}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(q_1)}) \omega_2(v_{\sigma(q_1+1)}, \ldots, v_{\sigma(q_1+q_2)})$$
where $\sum_k$ *is the group of permutation of* $\{1, \ldots k\}$ *and* $\epsilon(\sigma) \in \{\pm 1\}$ *is the sign of permutation.*

Proposition 1. *Assume* $\varphi_1, \ldots, \varphi_q \in V^*$
Then, $\varphi_1 \wedge \ldots \wedge \varphi_q(v_1, \ldots, v_q) = \dfrac{1}{q!} \sum_{\sigma \in \sum_q} \epsilon(\sigma) \prod_{i=1}^{q} \varphi_i(v_{\sigma(i)}) = \dfrac{1}{q!} det(\varphi_i(v_j))$
where, $\varphi_i(v_j)$ *denotes the matrix,*
$$\begin{pmatrix} \varphi_1(v_1) & \ldots & \varphi_1(v_q) \\ \vdots & \ddots & \vdots \\ \varphi_q(v_1) & \ldots & \varphi_q(v_q) \end{pmatrix}.$$

Definition 2.7. *A* **graded vector space** $V^\bullet$ *is a vector space containing subspaces* V^q*,* $q \in \mathbb{Z}$ *such that* $V^\bullet := \bigoplus_q V^q$*.*
An element $v \in V^q$ *is a* **homogeneous element** *of* **degree q**.

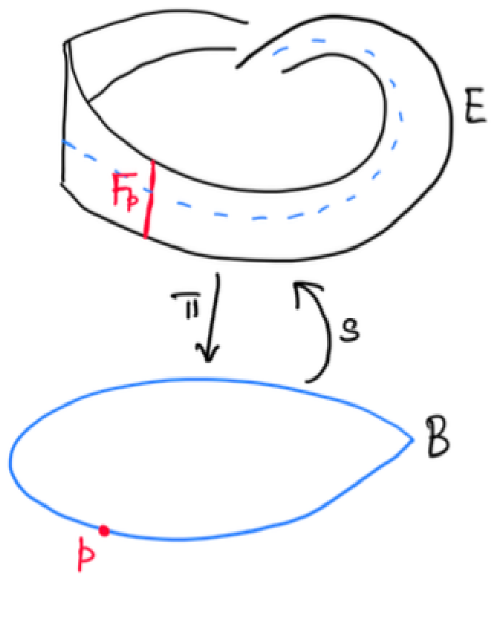
Definition 2.8. *The* **exterior algebra** *or* **Grassmann algebra** *is the graded algebra* $\Lambda^\bullet V^* := \bigoplus_{q \geq 0} \Lambda^q V^*$ *considered with the internal product given by the wedge product.*

2.2 Fibre Bundles

Definition 2.9. *Let* F *and* B *be topological spaces.*
A **fibre bundle** *over a* **base** B *with fibre* F *is a pair* (E, π) *where* E *is a topological space, the* **total space**, *and* $\pi \colon E \to B$ *is a continuous map, the* **projection**, *such that there exists an open cover* $\{U_\alpha\}_{\alpha \in \mathcal{I}}$*, and homeomorphism* $\varphi_\alpha \colon E\mid_{U_\alpha} := \pi^1(U_\alpha) \to U_\alpha \times F$ *such that the diagram commutes.*

$$\begin{tikzcd}
\pi^{-1}U_\alpha =: E\mid_{U_\alpha} \arrow[r, "\varphi_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times F \arrow[d, "\pi_1"] \\
& U_\alpha
\end{tikzcd}$$

The set $\{\varphi_\alpha \colon E\mid_{U\alpha} \to U_\alpha \times F\}_{\alpha \in \mathcal{I}}$ is called a **trivialization** of the bundle.

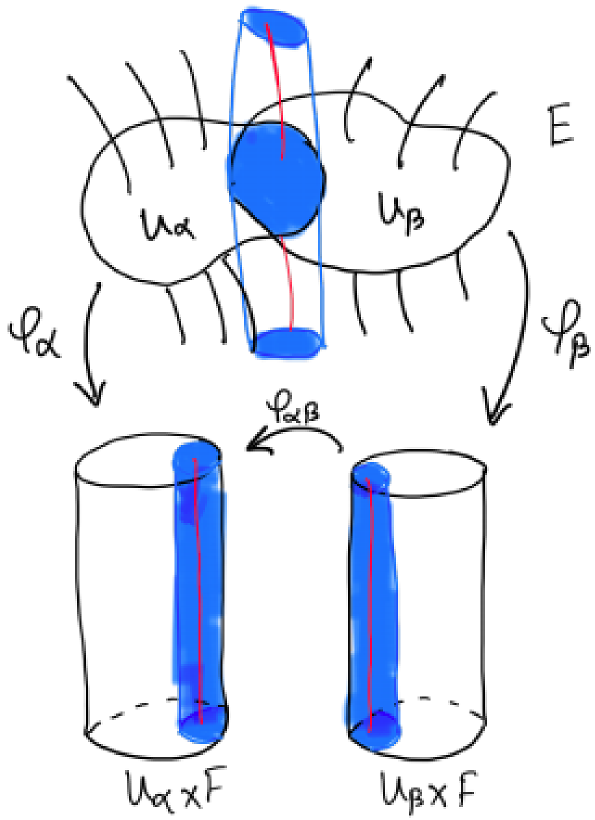


Example 14. *The* **trivial bundle** *is the product of topological spaces, with the projection on one factor:* $E := B \times F$, $\pi = \pi_1 \colon B \times F \to B$ *the map* $\pi(b, f) = b$.

Another example from the theory of topological coverings,

Example 15. *Fix* $d \in \mathbb{N}$ *and take* $E = B = \mathbb{S}^1 := \{(\cos\theta, \sin\theta)\} \subset \mathbb{R}^2$ *and* $\pi \colon E \to B$ *defined by* $\pi(\cos\theta, \sin\theta) = (\cos d\theta, \sin d\theta)$. *This is a fibre bundle with fibre a discrete set of cardinality* d.

Definition 2.10. *For every ordered pair of charts* $(U_\alpha, \varphi_\alpha)$ *and* (U_β, φ_β), *the set of* $\{\varphi_{\alpha\beta}\}$ *is the set of* **transition functions** *of the fibre bundles,* $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} = (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$.
They are of the form $\varphi_{\alpha\beta}(p, f) = (p, g_{\alpha\beta}(p)(f))$ *for some transition function* $g_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \to Aut(F)$.



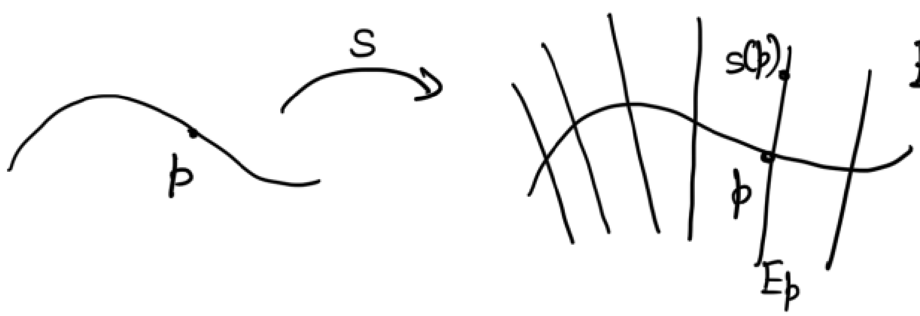
The transition function $g_{\alpha\beta}$ satisfies the following **cocylce conditions**:

1. $\forall \alpha \in \mathcal{I}, \forall p \in U_\alpha$, $g_{\alpha\alpha}(p) = Id_F$

2. $\forall \alpha, \beta \in \mathcal{I}, \forall p \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(p) = g_{\beta\alpha}^{-1}(p)$

3. $\forall \alpha, \beta, \gamma \in \mathcal{I}, \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$

Definition 2.11. *Let $\pi \colon E \to B$ be a fibre bundle. A* **section** *of E is a continuous map $s \colon B \to E$ such that $\pi \circ s = Id_B$.*



Example 16. *The* **zero section** *is a mapping $s \colon B \to E$ such that $s(p) = 0 \in E_p$ for all $p \in B$.*

Definition 2.12. *Let G be a subgroup of $Aut(F)$. A* **G-bundle** *is a fibre bundle with fibre F admitting a trivialization whose cocycle is contained in G:*
$\forall \alpha, \beta \in \mathcal{I}, \forall p \in U_\alpha \cap U\beta, g_{\alpha\beta}(p) \in G$.

2.3 Vector Bundles and Tangent Bundles

Definition 2.13. *A* **real vector bundle** *over B of rank r is a $G-$bundle with fibre $\mathbb{R}^r$ where G is the group of the invertible linear applications $GL(\mathbb{R}^r)$. A* **line bundle** *is a vector bundle of rank* 1*.*

Example 17. *The* **trivial vector bundle** *is the product of topological spaces, with the projection on one factor: $E \coloneqq B \times V$, $\pi = \pi_1 \colon B \times V \to B$ the map $\pi(b, f) = b$, where V is a vector space.*

Proposition 2. *Let B be a manifold, let $\mathcal{U} \coloneqq \{U_\alpha\}_\alpha \in \mathcal{I}$ be an open cover of B, $r \in \mathbb{N}$. Assume we have $\forall \alpha, \beta \in \mathcal{I}$, a smooth map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(\mathbb{R}^r)$ such that*

1. *$\forall \alpha \in \mathcal{I}$, $\forall p \in U$, $g_{\alpha\alpha}(p) = Id$*

2. *$\forall \alpha, \beta \in \mathcal{I}$, $\forall p \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(p) = g_{\beta\alpha}^{-1}(p)$*

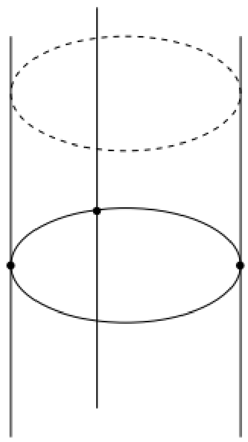
3. $\forall \alpha, \beta, \gamma \in I$, $\forall p \in U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta}(p) g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$

Then there is a unique, up to isomorphisms, real vector bundle E of rank r over B having a trivialization with cocycle $\{g_{\alpha\beta}\}$. Moreover E has a natural structure of manifold such that the projection $\pi\colon E \to B$ and the zero section $s_0\colon B \to E$ are smooth.
Moreover $dimE = dimB + rankE = dimB + r$, the differential of π is surjective at every point and the differential of s_0 is injective at every point.

We can now define the tangent bundle $\pi\colon T\mathcal{M} \to \mathcal{M}$ through its cocycle.

Definition 2.14. *Let $\mathcal{M}$ be a manifold of dimension n. Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{I}}$. Then the* **tangent bundle** *$\pi\colon T\mathcal{M} \to \mathcal{M}$ is the vector bundle of rank n given by the cocycle $g_{\alpha\beta}(p) = J(\varphi_{\alpha\beta})\varphi_\beta(p)$, where J is a Jacobi matrix.*

Example 18. *For $M = \mathbb{S}^1$, the tangent bundle looks like*



Definition 2.15. *A* **vector field** *v of a manifold $\mathcal{M}$ is a section of the tangent bundle $v\colon \mathcal{M} \to T\mathcal{M}$.*
A vector field is **smooth** *if it smooth among the map of manifolds.*
The smooth vector fields forms the vector space $\mathfrak{X}(\mathcal{M})$.

2.4 Differential Forms

We define the differential forms as sections of suitable vector bundles.

For every real manifold $\mathcal{M}$, the tangent bundle $T\mathcal{M}$, which induces $\forall 1 \leq q \leq dim\mathcal{M}$, by the theory of the vector bundles, a bundle $\Lambda^q T^*\mathcal{M} := \Lambda^q(T\mathcal{M})^*$.

Conventionally, we set $\Lambda^0 T^*\mathcal{M}$ to be the **trivial bundle of rank 1**.
The bundle $\Lambda^1 T^*\mathcal{M}$ is the **cotangent bundle**.
The bundle $\Lambda^{dim\mathcal{M}} T^*\mathcal{M}$ is the **canonical bundle**.

Definition 2.16. *A* **differential q-form** *on a manifold $\mathcal{M}$ is a map $\omega\colon \mathcal{M} \to \Lambda^q T^*\mathcal{M}$ such that $\pi \circ \omega = Id_{\mathcal{M}}$. The form is* **smooth** *if it is smooth as a map among manifolds. The smooth q-forms form a vector space $\Omega^q(\mathcal{M})$.*
Conventionally, $\Omega^0(\mathcal{M}) = \ell^\infty(\mathcal{M})$, $\Omega^q(\mathcal{M}) = \{0\}$ for $q < 0$ or $q > dim\mathcal{M}$, $\Omega^\bullet(\mathcal{M}) = \bigoplus_{q\in\mathbb{Z}} \Omega^q(\mathcal{M})$.

Example 19. *A $0-$form is a section of $\Lambda^0 T^*$ which by convention is just a smooth function.*

Example 20. *A $1-$form is a section of the cotangent bundle $\Lambda^1 T^*$.*

The q-forms act on $\mathfrak{X}(\mathcal{M})^q$; that is we can see every q-form ω as a map $\omega\colon \mathfrak{X}(\mathcal{M})^q \to \mathcal{C}^\infty(\mathcal{M})$ as follow:
For every choice of q smooth vector fields $v_1, \ldots, v_q, \omega(v_1, \ldots, v_q)$ is the function defined by $\forall p, \omega(v_1, \ldots, v_q)(p) \coloneqq \omega_p(v_1(p), \ldots, v_q(p))$.

Example 21. *Consider $\mathcal{M} = \mathbb{R}^3$, and use (x, y, z) to denote the canonical coordinates of $\mathcal{M}$*

1. *For a $\mathcal{C}^\infty$ function f,*
   $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
2. *For $1-$form $\omega = adx + bdy + cdz$,*
   we have,
   $$\begin{aligned} d\omega &= (\frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz) \wedge dx + (\frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial y}dy + \frac{\partial b}{\partial z}dz) \wedge dy \\ &\quad +(\frac{\partial c}{\partial x}dx + \frac{\partial c}{\partial y}dy + \frac{\partial c}{\partial z}dz) \wedge dz \\ &= (\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})dy \wedge dz + (\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x})dz \wedge dx + (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy \end{aligned}$$
3. *For a $2-$form,*
   $\omega = ady \wedge dz + bdz \wedge dx + cdx \wedge dy,$
   we have,
   $$d\omega = (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z})dx \wedge dy \wedge dz$$

3 de Rham Cohomology

We can now define the de Rham cohomology of a real manifold.

Let $\mathcal{M}$ be a manifold, or a disjoint union of manifolds. Consider the graded algebra $\Omega^\bullet(\mathcal{M}) \coloneqq \bigoplus_{q\in\mathbb{Z}} \Omega^q(\mathcal{M})$ and its exterior derivative (or differential), the operator of degree 1, $d\colon \Omega^\bullet(\mathcal{M}) \to \Omega^\bullet(\mathcal{M})$.

Definition 3.1. *Let $\omega \in \Omega^\bullet(\mathcal{M})$, then,*

1. *ω is* **closed** *if $d\omega = 0$, i.e. $\omega \in Kerd$*
2. *ω is* **exact** *if $\exists \eta \in \Omega^\bullet(\mathcal{M})$ such that $\omega = d\eta$, i.e. $\omega \in Imd$*

Example 22. *Let us consider the differential $1-$form*
$\omega = -\dfrac{y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy$
We claim that ω is a closed $1-$form but not a exact $1-$form. Infact let γ be a closed curve such that

$$\gamma\colon [0, 2\pi] \to \mathbb{R}^2$$
$$\theta \mapsto (\sin\theta, \cos\theta)$$

Computing the line integral,

$$\begin{aligned}\int_\gamma \omega &= \int_\gamma -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \\ &= \int_0^{2\pi} -\frac{\sin\theta}{\sin^2\theta+\cos^2\theta}(-\sin\theta)d\theta + \frac{\cos\theta}{\sin^2\theta+\cos^2\theta}(\cos\theta)d\theta \\ &= \int_0^{2\pi} dt \\ &= 2\pi\end{aligned}$$

Since, $\int \omega \neq 0$, ω is NOT exact $1-$form.

On the other hand if we compute $d\omega$, we have
$d\omega = dA \wedge dx + dB \wedge dy$,
where,
$A = -\dfrac{y}{x^2+y^2}$ *and* $B = \dfrac{x}{x^2+y^2}$

$$\begin{aligned}d\omega &= (\tfrac{\partial A}{\partial x}dx + \tfrac{\partial A}{\partial y}dy) \wedge dx + (\tfrac{\partial B}{\partial x}dx + \tfrac{\partial B}{\partial y}dy) \wedge dy \\ &= \frac{\partial A}{\partial y}dy \wedge dx + \frac{\partial B}{\partial x}dx \wedge dy \\ &= -\frac{\partial A}{\partial y}dx \wedge dy + \frac{\partial B}{\partial x}dx \wedge dy \\ &= (-\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x})dx \wedge dy \\ &= 0\end{aligned}$$

Thus ω is a closed $1-$form.

Proposition 3. *There is a unique operator, called* **exterior derivative** *or* **differential** $d\colon \Omega^\bullet(\mathcal{M}) \to \Omega^\bullet(\mathcal{M})$ *of degree* 1 *such that*

1. $\forall f \in \Omega^0(\mathcal{M}), \forall v \in \mathfrak{X}(\mathcal{M}), df(v) = v(f)$
2. $\forall q_1, q_2 \geq 0, \forall \omega \in \Omega^{q_1}(\mathcal{M}), \forall \omega \in \Omega^{q_2}(\mathcal{M}), d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge (-1)^{q_1}\omega_1 \wedge d\omega_2$
3. $d \circ d = 0$

If (U, φ) *is a chart with coordinates* $x_1, \ldots, x_n$ *and on* U
$\omega = \sum_{1\leq i_1 < \ldots < i_q \leq n} \omega_{i_1 \ldots i_q} dx_{i_1} \wedge \ldots \wedge dx_{i_q}$,
then,

$$\begin{aligned} d\omega &= \sum_{1\leq i_1 < \ldots < i_q \leq n} d\omega_{i_1 \ldots i_q} dx_{i_1} \wedge \ldots \wedge dx_{i_q}, \\ &= \sum_{1\leq i_1 < \ldots < i_q \leq n} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \ldots i_q}}{\partial x_i} dx_{i_1} \wedge \ldots \wedge dx_{i_q} \\ &= \varphi^* d((\varphi^{-1})^* \omega). \end{aligned}$$

By the property 2 of the last proposition, **every exact form is closed**.

Definition 3.2. *A* **differential complex** *is a pair* $(V^\bullet, d)$ *where* $V^\bullet = \bigoplus_{q\in\mathbb{Z}} V^q$ *is a graded vector space and* $d\colon V^\bullet \to V^\bullet$ *is an operator of degree* 1 *such that* $d \circ d = 0$.

The last definition implies that $(\Omega^\bullet(\mathcal{M}), d)$ is a differential complex.

Definition 3.3. *If* $(V^\bullet, d)$ *is a differential complex* $Imd \subset kerd$ *and we can define its* **cohomology** $H_d^\bullet(V^\bullet) := \dfrac{kerd}{Imd}$.
For every $\omega \in kerd$ *we denote by* $[\omega]$ *its class in* $H_d^\bullet(V^\bullet)$.
$H_d^\bullet(V^\bullet)$ *has a natural structure of graded vector space* $H_d^\bullet(V^\bullet) = \bigoplus_{q\in\mathbb{Z}} H_d^q(V^\bullet)$, *obtained by defining* $H_d^q(V^\bullet) := \{[\omega] \in H_d^\bullet(V^\bullet) \mid \omega \in V^q\}$.
In particular $H_d^q(V^\bullet) = \dfrac{kerd\mid_{V_q}}{dV_{q-1}}$.

Definition 3.4. *For every manifold (or disjoint union of manifolds)* $\mathcal{M}$ *the differential complex* $(\Omega^\bullet(\mathcal{M}), d)$ *is the* **de Rham complex** *of* $\mathcal{M}$.
Its cohomology is the **de Rham cohomology algebra** *or* **de Rham cohomology ring** *(sometimes denoted just by* **de Rham cohomology** *for short) of* $\mathcal{M}$, *the graded algebra*
$H_{DR}^\bullet(\mathcal{M}) = \dfrac{\textit{closed forms}}{\textit{exact forms}} = \bigoplus H_{DR}^q(\mathcal{M})$

where,
$H^q_{DR}(\mathcal{M}) = \dfrac{\textit{closed q-forms}}{\textit{exact q-forms}}$
is the q^{th} **de Rham cohomology group** *of* $\mathcal{M}$.
The algebra structure on $H^\bullet_{DR}(\mathcal{M})$ *is defined by the wedge product of de Rham cohomology classes* $[\omega_1] \wedge [\omega_2] \coloneqq [\omega_1 \wedge \omega_2]$.

NOTE: $H^q_{DR}(\mathcal{M})$, defined for all $q \in \mathbb{Z}$, equals $\{0\}$ unless $0 \leq q \leq dim\mathcal{M}$.

Example 23. *For the* $n-$*sphere,* $\mathbb{S}^n$*, and also when taken together with a product of open intervals, we have the following. Let* $n > 0$, $m \geq 0$*, and* $\mathbb{I}$ *be an open real interval. Then*
$$H^q_{DR}(\mathbb{S}^n \times \mathbb{I}^m) \simeq \begin{cases} \mathbb{R} & q = 0 \text{ or } q = n, \\ 0 & q \neq 0 \text{ and } q \neq n. \end{cases}$$

4 Conclusion

de Rham cohomology is an analytical way of approaching the algebraic topology of the manifold. It has been important in an enormous range of areas from algebraic geometry to theoretical physics. More refined use of analysis requires extra data on the manifold and we shall simply define and describe some basic features of Riemannian metrics. These generalize the first fundamental form of a surface and, in their Lorentzian guise, provide the substance of general relativity.

The importance of the theory is the introduction of the de Rham groups form a perfect example of the interaction between analysis and topology. For instance, if we know all about the differential forms of a manifold we can say something non-trivial about it's shape (is it diffeomorphic to a sphere, etc). Analogously, if we know about the shape of a manifold we can often conclude something relevant with respect to the functions on this manifold.

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