Fourier-Motzkin Elimination using Saturation Matrix

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Content

Contents

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$$\begin{array}{r} 3\,x_1 - 2\,x_2 + 1\,x_3 \leq 7 \\ -2\,x_1 + 2\,x_2 - 1\,x_3 \leq 12 \\ -4\,x_1 + 1\,x_2 - 3\,x_3 \leq 15 \end{array} \xrightarrow{1 \ step \ FME} \begin{array}{r} 0\,x_1 + 2\,x_2 - 1\,x_3 \leq 50 \\ 0\,x_1 - 5\,x_2 - 13\,x_3 \leq 73 \end{array}$$

Example

Eliminating t_1 from

$$A = \begin{cases} a_1 : 3t_1 - 2t_2 + t_3 \leq 7 \\ a_2 : -2t_1 + 2t_2 - t_3 \leq 12 \\ a_3 : -4t_1 + t_2 + 3t_3 \leq 15 \end{cases}$$

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$partition(A) = \{a_1\}, \{a_2, a_3\}$
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$$A'' = \begin{cases} -31t_3 \leq 396 \end{cases}$$

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- Jing, Moreno-Maza, Xie and Yuan [JMXY24] proposed a method using Saturation Matrix.

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- Users can access it using the following functions,
  1. `PolyhedralSets:-Project`
  2. `RegularChains:-FMXelim`
  3. `RegularChains:-SemiAlgebraicSetTools:-LinearSolve`

Definition (H-representation)

A polyhedral set P is any $\{\mathbf{x} \mid A\mathbf{x} <= \mathbf{b}\}$, where $A \in \mathbb{Q}^{m\times n}$ and $\mathbf{b} \in \mathbb{Q}^{m}$. Such a linear system is called an H-representation of P.

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- P is full-dimensional iff $A\mathbf{x} <= \mathbf{b}$ has no implicit equation.
- P is pointed, if A is full column rank.

NOTE: From now, P is full-dimensional and pointed.

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$$(a_2|b_1| + b_2a_1)x_2 + \cdots + (a_n|b_1| + b_na_1)x_n \leq |b_1|d_1 + a_1d_2.$$

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Definition

Having a linear inequality system S with m inequalities and n variables of the form $a_{i1}x_1 + \cdots + a_{in}x_n \leq d_i$, We can partition the inequalities in three groups with respect to x_1:

- A^+ set of inequalities with positive x_1 coefficient.
- A^- set of inequalities with negative x_1 coefficient.
- A^0 set of inequalities with zero x_1 coefficient.

Idea

Theorem

Let A' be the set formed by the combining each inequality in A^+ with each inequality in A^- and including inequalities in A^0 such that A' does not have x_1 term. Then,

$$(x_2, \cdots, x_n) \in Sol(A') \iff \exists x_1 \; (x_1, x_2, \cdots, x_n) \in Sol(A)$$

where $Sol(A)$ is a set of points satisfying all inequalities in A.

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- Combine inequalities in A^+ and A^- to form the resulting union.

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Improve Complexity

- FME's complexity is double exponential.
- Most of these generated inequalities are redundant.
- Detecting and removing them can significantly improve the complexity.

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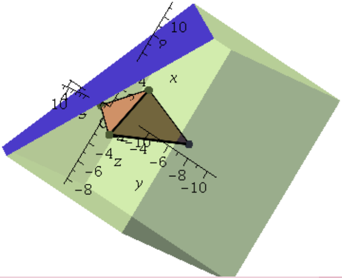
- Strongly redundant if $\mathbf{a}^t\mathbf{x} < \mathbf{b}$ for all $\mathbf{x} \in P$.
- Weakly redundant if $\mathbf{a}^t\mathbf{x} = \mathbf{b}$ holds for some $\mathbf{x} \in P$.



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- The most efficient variant, proposed by Fukuda and Prodon in [FP96] is implemented in the CDD library [cdd].

Given an inequality $A_i\mathbf{x} >= 0$, we partition the space $\mathbb{Q}^d$ into three regions,

- $H_i^+ = \{\mathbf{x} \in \mathbb{Q}^d \mid A_i\mathbf{x} > 0\}$.
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Let J be the set of column indices of R. The rays r_j ($j \in J$) are then partitioned as follows,

- $J^+ = \{j \in J \mid \mathbf{r_j} \in H_i^+\}$.
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Lemma

Let (A_K, R) be a DD pair and let i be a row index of A not in K. Then (A_{K+i}, R') is a DD pair, where R' is a $d \times |J'|$ matrix with column vectors $\mathbf{r_j}$ $(j \in J')$ defined by,

$$J' = J^+ \cap J^0 \cap (J^+ \times J^-), \text{ and}$$
$$\mathbf{r_{jj'}} = (A_i\mathbf{r_j})\mathbf{r_{j'}} - (A_i\mathbf{r_{j'}})\mathbf{r_j} \text{ for each } (j, j') \in J^+ \times J^-.$$

Let P be an H-cone defined by $P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} >= \mathbf{0}\}$.

Lemma (DD Pair Duality)

For any A and R, (A, R) is a DD pair $\iff$ (R^T, A^T) is a DD pair.

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Lemma (Adjacency Test)

Two distinct rays r and r' of the polyhedral cone P are called adjacent if they span a 2-dimensional face of P, that is,

$$Rank(A_{Z(r)\cap Z(r')}) = d - 2.$$

Algorithm 1 Double Description Method

Require: 1. a matrix $A \in \mathbb{Q}^{m \times n}$ defining the H-representation of a polyhedral cone P;
2. P is full-dimensional and pointed.

Ensure: a matrix R defining the V-representation of P.

Let K be the set of the indices of A's independent rows;
$R' := (A_K)^{-1}$;
Let J be the set of the columns of R';
while $K \neq \{1, \ldots, m\}$ do **do**
    Select a A-row index $i \notin K$;
    Set R' to empty matrix;
    $J^+, J^-, J^0 :=$ **Partition**(J, A_i);
    Append J^+ and J^0 as columns in R';
    for $p \in J^+$ **do**
        for $n \in J^-$ **do**
            if AdjacencyTest$(A_K, \mathbf{r_p}, \mathbf{r_n})$ **then**
                $\mathbf{r_{new}} := (A_i \mathbf{r_p})\mathbf{r_n} - (A_i \mathbf{r_n})\mathbf{r_p}$;
                Append $\mathbf{r_{new}}$ as a column to R';
            end if
        end for
    end for
    Let J be the set of the columns of R';
    $K = K \cup \{i\}$;
end while
Let R be the matrix created by the vectors in J as its columns;
return R;

Contents

Saturation Matrix

Definition

- A vertex $\mathbf{v} \in V$ of P *saturates* an inequality ℓ if $\mathbf{v}$ lies on the hyperplane $\mathcal{H}_\ell$, that is, if $\mathbf{a}^t\mathbf{v} = b$ holds.
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- $\mathcal{S}^{\mathcal{VR}}(\ell)$: all vertices and rays in $\mathcal{VR}(F)$ saturating ℓ.
- $\mathcal{S}^{\mathcal{H}}(\mathbf{u})$: all hyperplanes that $\mathbf{u}$ saturates.
- $\mathcal{S}^{\mathcal{H}}(\mathcal{S}^{\mathcal{VR}}(\ell)) = \bigcap_{\mathbf{u} \in \mathcal{S}^{\mathcal{VR}}(\ell)} \mathcal{S}^{\mathcal{H}}(\mathbf{u})$: inequalities saturated by all vertices or rays saturating ℓ.

Definition

The *saturation matrix* of F is the boolean matrix $\mathrm{satM}(F) \in \mathbb{Q}^{m \times k}$, where each entry (i, j) is defined as follows:

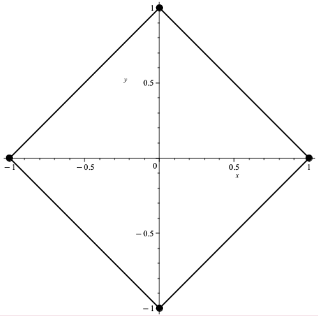
- 1 if the j-th element of $\mathcal{VR}(F)$ saturates the i-th inequality of F.
- 0 otherwise.

Example

F	$\mathcal{VR}(F)$
$\ell_1 : x + y \leq 1$	$\mathbf{v}_1 : (0, 1)$
$\ell_2 : -x - y \leq 1$	$\mathbf{v}_2 : (1, 0)$
$\ell_3 : x - y \leq 1$	$\mathbf{v}_3 : (-1, 0)$
$\ell_4 : -x + y \leq 1$	$\mathbf{v}_4 : (0, -1)$

satM(F)

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
ℓ_1	1	1	0	0
ℓ_2	0	0	1	1
ℓ_3	0	1	0	1
ℓ_4	1	0	1	0



From the saturation matrix $\mathrm{satM}(F) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, it is easy to obtain the following identities:

$$
\begin{aligned}
\mathcal{S}^{\mathcal{VR}}(\ell_1) &= \{\mathbf{v_1}, \mathbf{v_2}\} & \mathcal{S}^{\mathcal{H}}(\mathbf{v_1}) &= \{\ell_1, \ell_4\} \\
\mathcal{S}^{\mathcal{VR}}(\ell_2) &= \{\mathbf{v_3}, \mathbf{v_4}\} & \mathcal{S}^{\mathcal{H}}(\mathbf{v_2}) &= \{\ell_1, \ell_3\} \\
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\end{aligned}
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The saturation matrix is processed in two ways:

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  - Replace these columns with results of this bit-wise OR.

Contents

Algorithm 2 CheckRedundancy

Require: 1. the inequality system F with m inequalities;
2. the saturation matrix satM.

Ensure: the minimal system F_{irred} and the corresponding saturation matrix $\mathrm{satM}_{\mathrm{irred}}$.

$\mathit{Irredundant} := \{\mathrm{seq}(i, i = 1..m)\}$;
for i from 1 to m **do**
  if the number of nonzero elements in satM$[i]$ is less than n **then**
    $\mathit{Irredundant} := \mathit{Irredundant} \setminus \{i\}$;
    next;
  end if
  for j in $\mathit{Irredundant} \setminus \{i\}$ **do**
    if $\mathrm{satM}[i] = \mathrm{satM}[i] \mathit{AND} \mathrm{satM}[j]$ **then**
      $\mathit{Irredundant} := \mathit{Irredundant} \setminus \{i\}$;
      break;
    end if
  end for
end for
$F_{\mathrm{irred}} := [\mathrm{seq}(F[i], i \text{ in } \mathit{Irredundant})]$ and $\mathrm{satM}_{\mathrm{irred}} := [\mathrm{seq}(\mathrm{satM}[i], i \text{ in } \mathit{Irredundant})]$;
return F_{irred} and $\mathrm{satM}_{\mathrm{irred}}$;

Algorithm 3 Minimal projected representation

Require: 1. an inequality system F;
2. a variable order $x_1 > x_2 > \ldots > x_n$.

Ensure: the minimal projected representation *res* of F.

Compute the V-representation V of F by DD method;
Set $res := \text{table}()$;
Sort the elements in V w.r.t. the reverse lexicographic order;
Compute the saturation matrix satM;
$F, \text{satM} := \mathrm{CheckRedundancy}(F, \text{satM}(F))$;
$res[x_1] := F^{x_1}$;
for i from 1 to $n-1$ **do**
    $(F^p, F^n, F^0) := \text{partition}(F)$;
    $V_{new} := \text{proj}(V, \{x_i\})$;
    Merging: $\text{satM} := \text{Merge}(\text{satM})$;
    Let $F_{new} := F^0$ and $\text{satM}_{new} := \text{satM}[F^0]$;
    for each $f_p \in F^p$ and $f_n \in F^n$ **do**
        Append $\text{proj}((f_p, f_n), \{x_i\})$ to F_{new};
        Append $\text{satM}[f_p] AND \text{satM}[f_n]$ to satM_{new};
    end for
    $F, \text{satM} := \mathrm{CheckRedundancy}(F_{new}, \text{satM}_{new})$;
    $V := V_{new}$ and $res[x_{i+1}] := F^{x_{i+1}}$;
end for
return *res*;

Contents

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Given a H-representation $(A, \mathbf{b})$ with $A \in \mathbb{Q}^{m \times n}$, $\mathbf{b} \in \mathbb{Q}^m$ and $\mathrm{height}([A, \mathbf{b}]) = h$.

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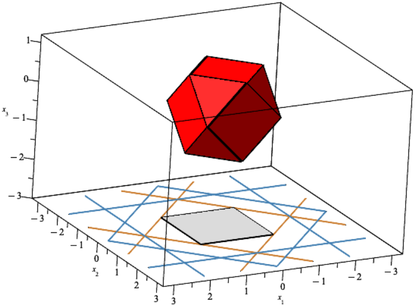
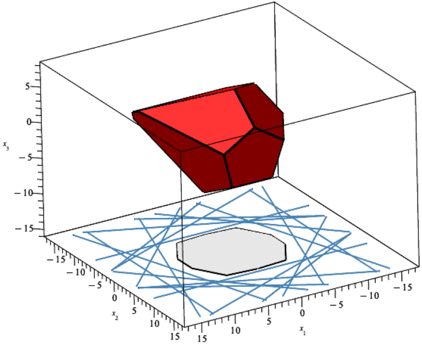
  - Detectiing redundancy for one inequality requires m^{n+1}; for all inequalities: m^{n+2} bit operations.

Cuboctahedron



1. strongly redundannt inequalities
2. weakly redundant inequalities eliminated by cardinality
3. weakly redundancies inequalities eliminated by containment

Snub disphenoid (triangular dodecahedron)



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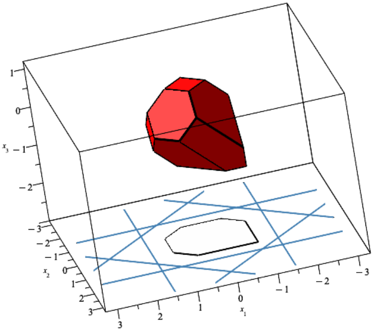
Truncated octahedron



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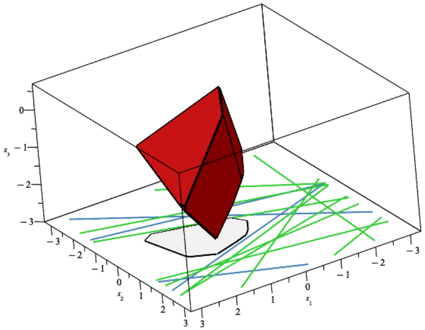
Random 3D polyhedron



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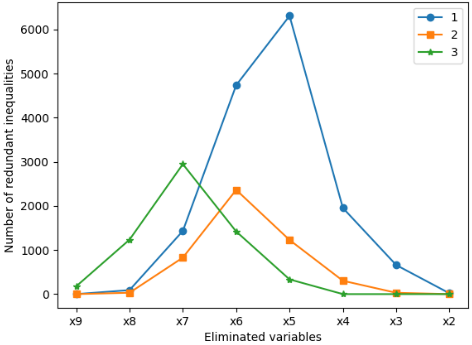
Random 10D polyhedron



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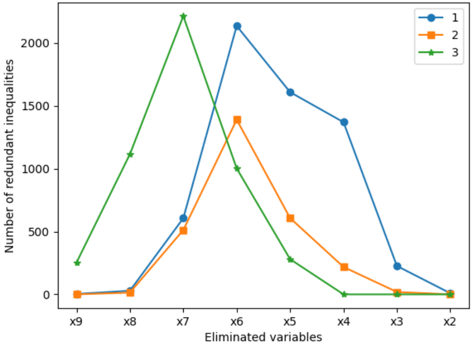

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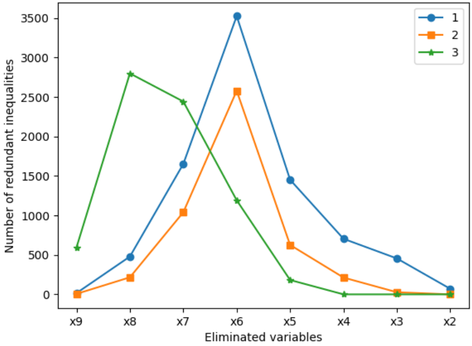

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- cddlib(by Fukuda [cdd]): Eliminates multiple variables at once, works with H-representation, with redundancy check via Linear Programming (LP).
- polylib(by Loechner [Lo99]): Eliminates multiple variables, works with V-representation, can convert between H- and V-representation as needed.

Each test case is associated with a triple (n, m, k) where n and m are the number of variables and inequalities in the input inequality system, while k is the number of vertices and rays.

test case	(n, m, k)	FME C	RTC	FME DD	FME VR	mpr	BPAS	cdd	polylib
32hedron	(6, 32, 11)	8.00	849.00	363.00	14576.00	6.54	16.80	4183.08	1.92
64hedron	(7,64,13)	25.00	3211.00	764.00	218833.00	13.05	52.42	>5min	1.67
francois	(13,27,2304)	2.00	101.00	48.00	65.00	499.92	253.66	388.36	> 5min
francois2	(13,31,384)	2.00	99.00	57.00	89.00	41.80	140.34	55.17	80.63
xavi	(2,7,5)	1.00	29.00	27.00	31.00	0.92	2.91	2.66	0.57
c6.in	(11,17,31)	6.00	29.00	903.00	>5min	9.85	12.72	84.11	5.56
c9.in	(16,18,140)	8.00	701.00	1685.00	>5min	25.08	65.54	151.17	131.53
c10.in	(18,20,142)	9.00	722.00	2038.00	>5min	22.10	98.68	249.02	16.06
e2.in	(6,9,8)	2.00	96.00	128.00	914.00	2.71	4.60	14.42	1.79
e7.in	(4,7,5)	1.00	28.00	46.00	75.00	1.92	3.12	5.24	1.09
e8.in	(3,6,4)	1.00	14.00	69.00	99.00	2.51	2.30	2.56	0.84
e13.in	(6,9,18)	2.00	91.00	133.00	962.00	4.30	3.74	13.13	1.35
e14.in	(5,7,10)	1.00	32.00	153.00	360.00	2.20	1.58	9.91	1.42
S24	(24, 25,25)	1802.00	1996.00	560.00	>5min	23.50	58.80	748.67	17.47
cube	(10, 20,1024)	1.00	63.00	50.00	70.00	81.33	201.92	125.900	161.06
C56	(5, 6,6)	2.00	216.00	91.00	1399.00	3.67	4.09	11.81	0.79
C68	(6, 16,8)	1.00	132.00	50.00	82.00	4.18	10.13	505.00	1.86
C1011	(10, 11,11)	13.00	>5min	920.00	>5min	24.99	115.68	1716.25	9.99
C510	(5, 42,10)	14.00	314.00	355.00	188.00	12.00	40.01	>5min	4.42
T1	(5, 10,38)	3.00	655.00	216.00	2819.00	5.61	16.44	27.42	8.81
T3	(10,12,29)	18.00	>5min	1943.00	>5min	21.29	141.64	288.07	12.07
T5	(5, 10,36)	4.00	1088.00	411.00	3812.00	8.12	15.62	22.92	4.76
T6	(10,20,390)	44901.00	>5min	>5min	>5min	1142.9	23800.11	14937.61	>5min
T7	(5, 8,26)	2.00	670.00	262.00	3559.00	5.81	10.79	13.96	4.00
T9	(10,12,36)	21.00	>5min	2501.00	>5min	36.56	414.53	479.18	100.34
T10	(6, 8,24)	6.00	1228.00	263.00	7190.00	4.58	13.65	18.39	5.27
T12	(5, 11,42)	9.00	959.00	1144.00	8950.00	8.52	19.03	38.65	8.60
R_15_20	(15, 20,1328)	27800.00	>5min	>5min	>5min	28430.40	336035.00	38037.21	>5min

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