POLISH SPACES

– Seminar Paper –

submitted by

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Preface

This note is part of my seminar on "Polish Spaces" that I delivered on the 12 and the 14 of November 2020.
Polish spaces are so named because they were first extensively studied by Polish topologists and logicians—Sierpiński, Kuratowski, Tarski and others. However, Polish spaces are mostly studied today because they are the primary setting for descriptive set theory, including the study of Borel equivalence relations. Polish spaces are also a convenient setting for more advanced measure theory, in particular in probability theory.

The contents of this notes are mostly based from the book of Alexander S. Kechris, "Classical Descriptive Set Theory" published by Springer in 1995 *(AK)*. We are essentially going to study: *definitions and examples of Polish spaces, countable products, G_δ subspaces and* [*AK, Theorem 3.11*]. *Sets with Baire property and Baire measurable function. The Baire category*: *meager and residual sets, Baire spaces, Baire category theorem* [*AK, Theorem 8.4*], *the Kuratowski-Ulam theorem* [*AK, Theorem 8.41*], *Baire measurable functions are continuous on a residual set* [*AK, Theorem 8.38*].
I have tried to provide a somewhat 'simplified' proofs of the results and covered almost all background knowledge that would be required to read this notes without any difficulty. I have also tried to make this notes error-free to the best of my ability. But if you notice any error or any inconsistency, please let me know.

Lastly, thank you for choosing to read my notes. I hope you enjoy it :)

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Contents

1 Introduction

1.1 Topological Spaces

Definition 1.1. *A* **topological space** *is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X, satisfying the following axioms:*

1. *The empty set ϕ and X itself belong to τ.*

2. *Any arbitrary (finite or infinite) union of members of τ still belongs to τ.*

3. *The intersection of any finite number of members of τ still belongs to τ.*

The elements of τ are called **open sets** *and the collection τ is called a* **topology** *on X.*

The complement of open sets are called **closed sets**. ϕ and X are both open and closed, otherwise known as **clopen sets**.

Definition 1.2. *In a topological space a* $\boldsymbol{G_\delta}$ **set** *is a countable intersection of open sets. And an* $\boldsymbol{F_\sigma}$ **set** *is a countable union of closed sets.*

Example 1. *Each open set is trivially a G_δ set.*

Example 2. *Similarly, each closed set is trivially an F_σ set.*

Example 3. *The irrational numbers can be written as $\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$ is a G_δ set.*

Example 4. *Similarly, the set of rationals $\mathbb{Q}$ is an F_σ set.*

Definition 1.3. *If X is a set, then a* **basis** *is a collection $\mathcal{B}$ of subsets of X called* **basis elements** *such that:*

1. *For each $x \in X$ there is at least one basis element B containing x.*

2. *If $x \in B_1 \cap B_2$, where B_1 and B_2 are basis elements then there is a basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies the above two conditions then the **topology τ generated by $\mathcal{B}$** is: A subset U of X is said to be open in X, if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

NOTE: Each basis element is itself an element in τ.

Definition 1.4. *A* **subbasis** *for the topology of* X *is a collection* $\mathcal{S} \subseteq \tau$ *with the property that the set of finite intersections of sets in* $\mathcal{S}$ *is a basis for the topology of* X*.*

Definition 1.5. *Let* $(X_i, \tau_i)_{i\in I}$ *be a family of topological spaces. The* **product topology** *on* $\prod_{i\in I} X_i$ *is the topology whose open sets are the unions of the products of the form* $\prod_{i\in I} U_i$*, where* $U_i \in \tau_i$ *for each* $i \in I$*, and* $U_i = X_i$ *except for finitely many* i*'s in* I*.*

The **basis for the product topology** is $B \coloneqq \{\prod_{i\in I} B_i \mid B_i \in X_i$ is a basis for the topology on $X_i\}$.

Example 5. *If one starts with the standard topology on the real line* $\mathbb{R}$ *and defines a topology on the product of* n *copies of* $\mathbb{R}$ *in this fashion, one obtains the ordinary* **Euclidean topology** *on* $\mathbb{R}^n$*.*

1.2 Metric Spaces

Definition 1.6. *Let* X *be a set, and* $d\colon X \times X \to [0, \infty)$ *be a function. We say that* d *is a* **metric** *on* X *if, for* $x, y, z \in X$ *which satisfies:*

1. $d(x, y) = 0$ *if and only if* $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

We say that (X, d) *is a* **metric space** *if* X *is a set and* d *is a metric on* X*.*

Example 6. $X = \mathbb{R}$*, equipped with* d *defined by* $d(x, y) := |x - y|$*, is a metric space.*

Definition 1.7. *An* **open ball with center** $x \in X$ **and radius** r *of a metric space* (X, d) *is* $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$*.*

Open balls form the basis of the topology of the metric spaces.

Definition 1.8. *A topological space* X *is called* **metrizable** *if there exists a metric* d *so that* τ *is the topology of* (X, d)*. In this case we say* d *is* **compatible** *with* τ*.*

NOTE: If τ is metrizable with compatible metric d, then $d' = \dfrac{d}{1+d}$ is also compatible.

Definition 1.9. *The* **product** *of a sequence of metric spaces* (X_n, d_n) *is the metric space* $(\prod_n X_n, d)$ *where the distance* $d(x,y) = \sum_{n=0}^{\infty} 2^{-(n+1)} \dfrac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$ *with* $x = (x_n)$ *and* $y = (y_n)$.

Definition 1.10. *Let* X *and* Y *be metric spaces with metrics* d_X *and* d_Y. *A map* $f\colon X \to Y$ *is called an* **isometry** *or* **distance preserving** *if for any* $a, b \in X$ *one has* $d_Y(f(a), f(b)) = d_x(a, b)$.

Definition 1.11. *A topological space* X *is* **second countable** *if there is a countable basis for its topology.*

Example 7. *Let* $n \in \mathbb{N}$. *Consider the Euclidean space* $\mathbb{R}^n$ *with its Euclidean metric topology. Then* $\mathbb{R}^n$ *is second countable.*

1.3 Separation

Definition 1.12. *If* X *is a topological space, we say* X *is* $\boldsymbol{T_1}$ *if every singleton of* X *is closed.*

Example 8. *The empty space and one point space are* T_1, *because the condition for two points is vacuously satisfied.*

Example 9. *More generally, any finite discrete space – a finite topological space where all subsets are open is* T_1.

Example 10. *All one point space in an Euclidean space is closed, hence is* $T_!$.

Definition 1.13. *If* X *is a topological space, we say* X *is* **regular** *if, for any* x *in* X *and any open neighborhood* N *of* x, *there is an open neighborhood* U *of* x *such that* $\overline{U} \subseteq N$.

Theorem 1 (Urysohn Metrization Theorem). *Let,* X *be a second countable topological space. Then* X *is metrizable iff* X *is* T_1 *and regular.*

Definition 1.14. *A subset* $D \subseteq X$ *of a topological space is* **dense** *if it meets every non-empty open sets.*

Definition 1.15. *A topological space X is* **separable** *if it has a countable dense subset.*

Example 11. $\mathbb{R}, \mathbb{C}$ *with their standard topology are separable metric spaces.*

Theorem 2. *A second countable space is separable.*

Proof. Let (X, τ) be a second countable topological space. Then there exists a countable basis $\mathcal{B} = \{B_1, B_2, \ldots, B_n \ldots\}$ of τ. Since $\mathcal{B}$ is a basis of τ we have that every open set $U \in \tau$ can be expressed as the union of sets in some subcollection $\mathcal{B}^* \subset \mathcal{B}$. In particular $U = \bigcup_{B \in \mathcal{B}*} B$.
We must now construct a countable dense subset of X. Assume that $\mathcal{B}$ does not contain the empty set. If it does contain the empty set then we can discard it. Then for each $B_n \in \mathcal{B}$ take $x \in B_n$ and define the set A as: $A = \{x_n \mid x_n$ is any element in $B_n, n = 1, 2, \ldots\}$.
Then, A is a countable subset of X since we take one element from each set in the countable basis.
Furthermore, for all $U \in \tau \setminus \{\phi\}$ we have that $A \cap U \neq \phi$ because A contains one element from each of the basis sets and U is the union of some subcollection of the basis sets. Therefore A is a dense subset of X.
Hence, A is a countable dense subset of X, so (X, τ) is a separable topological space. □

NOTE: The converse of the above theorem is not true in general.

Example 12. *If X is a finite set and τ is the discrete topology on X then (X, τ) is a separable topological space.*

Let X be a finite set with n elements. Then $X = \{a_1, a_2, \ldots, a_n\}$. If τ is the discrete topology on X then $\tau = \mathcal{P}(X)$. Clearly every subset of X is countable since X is a finite set, so we only need to find a dense subset of X.
Take $A = X$. Then for all $U \in \tau \setminus \{\phi\}$ we have that $U \subseteq A$. So $A \cap U \neq \phi$. Hence, A is a countable and dense subset of X so (X, τ) is a separable topological space.

Example 13. *If X is a countable set and τ is the discrete topology on X then (X, τ) is a separable topological space.*

If X is countable then X is either finite or countably infinite. Example 12 shows that if X is finite and τ is the discrete topology on X then (X, τ) is a separable

topological space. We will now show that if X is countably infinite then (X,τ) is also a separable topological space.
If X is countably infinite then: $X = \{a_1, a_2, \dots\}$
If we let $A = X$ then every $U \in \tau \setminus \{\phi\}$ is such that $U \subseteq A$ so $A \cap U \neq \phi$. Therefore A is a countable (and dense) subset of X, so (X,τ) is separable.

Theorem 3 (**Urysohn's Lemma**). *Let X be a metrizable space. If A and B are two disjoint closed subsets of X, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

Theorem 4 (**Tietze Extension Theorem**). *Let X be a metrizable space. If $A \subseteq X$ is closed and $f\colon A \to \mathbb{R}$ is continuous, there is a function $\hat{f}\colon X \to \mathbb{R}$ which is continuous and extends f. Moreover, if f is bounded by M i.e. $|f(x)| \leq M$ for all $x \in A$ then so is $\hat{f}$.*

2 Polish Spaces

2.1 Properties

Let (X, d) be a metric space. A sequence (x_n) of points of X is called a **Cauchy sequence** if for every positive real number $\epsilon > 0$ there is a positive integer N such that for all positive integers $m, n > N$ the distance $d(x_n, x_m) < \epsilon$.

Definition 2.1. *A metric space (X, d) in which every Cauchy sequence converges to an element of X is called* **complete***.*

Example 14. *The real or complex numbers with the standard metric is a complete metric space.*

Example 15. *Finite dimensional real and complex vector spaces are are complete.*

Example 16. *The space of square integrable function on the unit interval $\mathcal{L}^2([0,1])$ is complete.*

Definition 2.2. *Given a metric space (X, d) there is a complete metric space $(\hat{X}, \hat{d}))$ such that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and X is dense in $\hat{X}$. This space is unique upto isometry and called* **completion** *of (X, d).*

Theorem 5. *$\hat{X}$ is separable iff X is separable.*

Definition 2.3. *A topological space (X, τ) is called* **completely metrizable** *if it admits a compatible metric d such that (X, d) is complete.*

Definition 2.4 (Polish space). *A topological space (X, τ) is called Polish if it is separable and completely metrizable.*

The completion of a separable metric space is Polish.

Example 17. *$\mathbb{R}, \mathbb{C}$ equipped with their standard topology are Polish spaces.*

Theorem 6. *The class of:*

1. *completely metrizable spaces is closed under countable products and topological sums,*
2. *Polish spaces is closed under countable products and countable topological sums.*

Proof. Let $(X_i)i \in I$ be a sequence of completely metrizable spaces. Then:

1. We saw that $\prod_{i \in I} X_i$ is metrizable if I is countable. Assume that the X_i's are completely metrizable. The proof of this fact shows that if $((x_i^n)_{i \in I})_{n \in \mathbb{N}}$ is Cauchy, then $(x_i^n)_{n \in \mathbb{N}}$ is Cauchy for each $i \in I$, so that $(x_i^n)_{n \in \mathbb{N}}$ converges to $x_i \in X_i$. Now $((x_i^n)_{i \in I})_{n \in \mathbb{N}}$ converges to $(x_i)_{i \in I}$. Thus, $\prod_{i \in I} X_i$ is completely metrizable. Let d_i be a metric on X_i defining its topology. We set $d((i,x),(j,y)) \coloneqq d_i(x,y)$ if $i = j$, 1 otherwise. Then d is a metric on $\bigoplus_{i \in I} X_i$ defining its topology, and we can check that it is complete.

2. We apply (1) and the fact that the class of separable spaces is closed under countable products and countable topological sums.

□

Example 18. $\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}}$ *with the product topology are Polish.*

Example 19. *The open unit interval* $(0,1)$*, the* **unit interval** $\mathbb{I} = [0,1]$*, the* **unit circle** S^1*,* $\mathbb{R} \setminus \mathbb{Q}$*, the* **n-dimensional cube** $\mathbb{I}^n$*, the* **Hilbert cube** $\mathbb{I}^{\mathbb{N}}$*, the* **n-dimensional torus** $\mathbb{T}^n \coloneqq S^1 \times \ldots \times S^1$ *(n-copies), the* **infinite dimensional torus** $\mathbb{T}^{\mathbb{N}}$ *are Polish.*

Example 20. *Any set A with discrete topology is completely metrizabe and if it is countable, it is Polish. Hence, $A^{\mathbb{N}}$ is also Polish. In particular, the* **Cantor space** $\mathcal{C} \coloneqq \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$ *and the* **Baire space** $\mathcal{N} \coloneqq \mathbb{N}^{\mathbb{N}}$ *are Polish spaces with the product topology of the discrete topologies.*

Example 21. *The topology of any (real or complex)* **Banach space** *is completely metrizable, and for separable Banach Spaces, it is Polish.*

2.2 Kuratowski Theorem

Theorem 7. *Let X be a metrizable space, and A be a closed subset of X. Then A is G_δ.*

Proof. Let d be a metric defining the topology of X. Note that $A = \bigcap_{n \in \mathbb{N}} \{x \in X \mid d(x,A) < \frac{1}{n+1}\}$, so we are done since $x \to d(x,A)$ is continuous. □

Definition 2.5. *Let, (X,d) be a metric space and $A \subseteq X$, then the* **diameter** *of A is $diam(A) \coloneqq sup\{d(x,y) \mid x,y \in A\}$.*

Definition 2.6. *Let,* (X, τ) *be a topological space and* (Y, d) *be a metric space, and let* $f: A \to Y$ *where* $A \subseteq X$*, then the* **oscillation of** f **at** $x \in X$*,* $osc_f(x) := inf\{diam(f(U \cap A) \mid U \text{ is an open neighbourhood of } x \in X)\}$*.*

NOTE: If $x \in A$ then x is a continuity point of f iff $osc_f(x) = 0$.

Theorem 8 (Kuratowski). *Let* X *be a metrizable space,* Y *be a completely metrizable space,* $A \subseteq X$ *and* $f: A \to Y$ *be a continuous function. Then we can find a* G_δ *subset* G *of* X *with* $A \subseteq G \subseteq \overline{A}$ *and* $g: G \to Y$ *continuous extension of* f*.*

Proof. We set $G := \overline{A} \cap \{x \in X \mid osc_f(x) = 0\}$. Note that $osc_f(x) = 0 \iff \forall n \in \mathbb{N}$ there is a neighbourhood U of x with diameter $diam(f(U \cap A)) < \frac{1}{n+1}$, so that $\{x \in X \mid osc_f(x) = 0\}$ is a G_δ set, as well as the closed set $\overline{A}$, by Theorem 7. Thus G is G_δ and contained in $\overline{A}$. If $x \in A$, then $x \in \overline{A}$ and $osc_f(x) = 0$ since f is continuous.
Now, let $x \in G$. As $x \in A$, there is a sequence (x_n) of points of A converging to x. Then $diam(f(\{x_{n+1}, x_{n+2}, \ldots\}))_{n\in\mathbb{N}}$ converges to 0, so that the sequence $f(x_n)$ is Cauchy and thus converges to $g(x) \in Y$. Note that g is well defined, and extends f. In order to see that g is continuous, we need to check that $osc_g(x) = 0$ for each $x \in G$. If U is open in X, then $g(U \cap G) \subseteq f(U \cap A)$, so $diam(g(Y \cap G)) \leq diam(f(U \cap A))$ and $osc_g(x) = osc_f(x) = 0$. □

2.3 Polish Subspaces

Theorem 9. *If* X *is metrizable and* $Y \subseteq X$ *is completely metrizable, then* Y *is a* G_δ *set in* X*. Conversely, if* X *is a completely metrizable space and* $Y \subseteq X$ *is a* G_δ *set, then* Y *is completely metrizable.*
In particular, **a subspace of Polish space is Polish iff it is a** $\mathbf{G_\delta}$ **set***.*

Proof. The first assertion follows from Kurtowski's Theorem. Consider the identity function on Y, $Id_Y: Y \to Y$. It is continuous, so there is a G_δ set call it G, with $Y \subseteq G \subseteq \overline{Y}$. There is also a continuous extension $g: G \to Y$ of Id_Y. Since, Y is dense in G, so $g = Id_G$. Hence, $Y = G$.
Conversely, let $Y = \bigcap_n U_n$, where U_n is open subset of X. Let $F_n = X \setminus U_n$. Let, d be a complete compatible metric on X. Define a new metric d' on X as
$$d' := d(x, y) + \sum_{n=0}^{\infty} \min \{2^{-(n+1)}, \left|\frac{1}{1 - d(x, F_n)} - \frac{1}{1 - d(y, F_n)}\right|\}.$$
This metric is compatible with the topology of Y. We show that (Y, d) is complete.

Let, (y_i) be a Cauchy sequence in (Y, d'). Then, it is Cauchy in (X, d). So, $y_i \to y$ in X. But, also for each n, $lim_{i,j\to\infty} \left| \frac{1}{1-d(x, F_n)} - \frac{1}{1-d(y, F_n)} \right| = 0$. So, for each n, $\frac{1}{d(y_i, F_n)}$ converges in $\mathbb{R}$, so $d(y_i, F_n)$ is bounded away from 0. Since, $d(y_i, F_n) \to d(y, F_n)$, we have $d(y, F_n) \neq 0$ for all n, so $y \notin F_n$ for all n, i.e., $y \in Y$. Clearly, $y_i \to y$ in (Y, d'). $\square$

3 Baire Categoty

3.1 Meager Sets

Definition 3.1. *Let (X, τ) be a topological space. We say that $A \subseteq X$ is* **nowhere dense** *if its closure $\overline{A}$ has empty interior $Int(\overline{A}) = \phi$*

Example 22. *The boundary of every open set and of every closed set is nowhere dense.*

Example 23. *The cantor set $\mathcal{C}$ is nowhere dense.*

Example 24. *In a T_1 space, any singleton set that is not an isolated point is nowhere dense.*

Example 25. *$\mathbb{Z}$ is nowhere dense in $\mathbb{R}$ and $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.*

Example 26. *The set $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$.*

Definition 3.2. *Let (X, τ) be a topological space. We say that $A \subseteq X$ is* **meager** *(or* **of the first category***) if it is a countable union of nowhere dense set, i.e. $A = \bigcup_{n \in \mathbb{N}} \{A_n \mid Int(\overline{A_n}) = \phi\}$.*

A non-meager set is also called a set **of the second category**.

Example 27. *A singleton set is always a non-meager subspace.*

Example 28. *The cantor set $\mathcal{C}$ is meager in $\mathbb{R}$ because it is closed and has empty interior.*

Example 29. *An countable Hausdorff space without isolated points is meager.*

Example 30. *Any topological space that contains an isolated point is non-meager.*

Example 31. *The set $\mathbb{Q}$ is meager in $\mathbb{R}$ because it is enumerable, so that we can write $\mathbb{Q} = \{q_1, q_2, \ldots\} = \bigcup_{i=1}^{\infty} \{q_i\}$ where $\{q_i\}$ is nowhere dense for all $i \in \mathbb{N}$.*

Definition 3.3. *Let (X, τ) be a topological space. We say that $A \subseteq X$ is* **comeager** *(or* **residual***) if its complement is meager.*

A set is residual iff it contains the intersection of a countable family of dense open sets.

Example 32. *$\mathbb{Z}$ is residual in $\mathbb{Z}$ because in $\mathbb{Z}$ every set is open.*

3.2 Baire Space

Definition 3.4. *A topological space (X, τ) is called a* **Baire space** *if it satisfies the following equivalent conditions:*

1. *Every non-empty open set in X is non-meager.*
2. *Every residual set in X is dense.*
3. *The intersection of countably many dense open sets in X is dense.*

Example 33. *The space $\mathbb{R}$ of real numbers with the standard topology, is a Baire space.*

Example 34. *The Cantor set $\mathcal{C}$ is a Baire space.*

Example 35. *The space of rational numbers $\mathbb{Q}$ with the standard topology inherited from the reals $\mathbb{R}$ is not a Baire space, since it is the union of countably many closed sets without interior, the singletons.*

Theorem 10 (**Baire Category Theorem**)**.** *Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.*

Proof. Let X be a completely metrizable space and let d be a complete metric on X compatible with the topology. Suppose that A_n are dense open subsets of X. To show that $\bigcap_{n\in\mathbb{N}} A_n$ is dense it suffices to show that for any nonempty open subset A of X, $\bigcap_{n\in N}(A \cap A_n) = A \cap \bigcap_{n\in\mathbb{N}} A_n \neq \phi$.
Because, A is a nonempty open set it contains an open ball B_1 of radius < 1 with $\overline{B_1} \subset A_1$. Since, A_1 is dense and B_1 is open, $B_1 \cap A_1 \neq \phi$ and is open because both B_1 and A_1 are open. As $B_1 \cap A_1$ is a nonempty open set it contains an open ball B_2 of radius $< \frac{1}{2}$ with $\overline{B_2} \subset B_1 \cap A_1$. Suppose that $n > 1$ and that B_n is an open ball of radius $< \frac{1}{n}$ with $\overline{B_n} \subset B_{n-1} \cap A_{n-1}$. Since A_n is dense and B_n is open, $B_n \subset A_n \neq \phi$ and is open because both B_n and A_n are open. As $B_n \subset A_n$ is a nonempty open set it contains an open ball B_{n+1} of radius $< \frac{1}{n+1}$ with $\overline{B_{n+1}} \subset B_n \subset A_n$. Then, we have $B_{n+1} \subset B_n$ for each $n \in \mathbb{N}$. Letting x_i be the center of B_i, we have $d(x_j, x_i) < \frac{1}{i}$ for $j > i$, and hence x_i is a Cauchy sequence. Since (X, d) is a complete metric space, there is some $x \in X$ such that $x_i \to x$. For any m there is some i_0 such that $i \geq i_0$ implies that $d(x_i, x) < \frac{1}{m}$, and hence $x \in B_m = \bigcap_{n=1}^{m} B_n$. Therefore, $x \in \bigcap_{n\in\mathbb{N}} B_n \subset_{n\in\mathbb{N}} (A \cap A_n)$, which shows that $\bigcap_{n\in\mathbb{N}} A_n$ is dense and hence that X is a Baire space.

Let X be a locally compact Hausdorff space. Suppose that A_n are dense open subsets of X and that A is a nonempty open set. Let $x_1 \in A$, and because X is a locally compact Hausdorff space there is an open neighborhood V_1 of x_1 with $\overline{V_1}$ compact and $\overline{V_1} \subset A$. Since A_1 is dense and A_1 is open, there is some $x_2 \in V_1 \cap A_1$. As $V_1 \cap A_1$ is open, there is an open neighborhood V_2 of x_2 with $\overline{V_2}$ compact and $\overline{V_2} \subset V_1 \subset A_1$. Thus, $\overline{V_n}$ are compact and satisfy $\overline{V_{n+1}} \subset \overline{V_n}$ for each n, and hence $\bigcap_{n\in\mathbb{N}} \overline{V_n} \neq \phi$. This intersection is contained in $\bigcap_{n\in\mathbb{N}}(A \cap A_n)$ which is therefore nonempty, showing that $\bigcap_{n\in\mathbb{N}} A_n$ is dense and hence that X is a Baire space. □

3.3 Baire Measurability

We now introduce a notion of regularity, being equal to an open set modulo a meager set.

Definition 3.5. *Let X be a set. Then a $\sigma-$**ideal** on X is a collection $\mathcal{I}$ of subsets of X such that:*

1. *If $A \subset B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$*
2. *If $A_1, A_2, \ldots \in \mathcal{I}$, then there exists B such that $B \in \mathcal{I}$ and $\bigcup_i A_i \subseteq B$*
3. *$\phi \in \mathcal{I}$*

Let, $\mathcal{I}$ be a $\sigma-$**ideal** on a set X. If $A, B \subseteq X$ we say that A, B are **equal modulo** $\mathcal{I}$, denoted as $A =_{\mathcal{I}} B$ if the **symmetric difference** $A\Delta B \coloneqq (A \setminus B) \cup (B \setminus A) \in \mathcal{I}$.

In particular if $\mathcal{I}$ is the $\sigma-$ideal of meager sets of topological space, we write $\mathbf{A} =^* \mathbf{B}$ if A and B are **equal modulo meager sets**.

Definition 3.6. *Let, (X, τ) be a topological space. A set $A \subseteq X$ has the* **Baire property** *(***BP***) if $A =^* U$ for some $U \subseteq X$.*

Theorem 11. *Let X be a topological space. The class of sets having Baire property is a $\sigma-$algebra on X. It is the smallest $\sigma-$algebra containing all open sets and meager sets.*

Proof. If U is open, then $\overline{U} \setminus U$ is closed nowhere dense and so is meager. Similarly, if F is closed, then $F \setminus Int(F)$ is closed nowhere dense. Thus, $U =^* \overline{U}$ and $F =^* Int(F)$.

Now, if A has the BP, so that $A =^* U$ for some open U, then $X \setminus A =^* X \setminus U =^*$

$Int(X \setminus U)$, so $X \setminus A$ has the BP. Finally, if each A_n has the BP, $A_n =^* U_n$, where U_n is open, then $\bigcup_n A_n =^* \bigcup U_n$, so $\bigcup_n A_n$ has the BP.
The last assertion follows from the fact that if $A =^* U$, where U is open, then with $M = A\Delta U$, M is meager, and $A = M\Delta U$. □

NOTE: All F_σ and G_δ sets have the BP but the converse is not true.

Theorem 12. *Let, X be a topological space and $A \subseteq X$. Then the following statements are equivalent:*

1. *A has the BP*
2. *$A = G \cup M$ where G is G_δ and M is meager set*
3. *$A = F \setminus M$ where F is F_σ and M is meager set*

Proof. By Theorem 11, (2) $\implies$ (1) and (3) $\implies$ (1).

To show (1) $\implies$ (2) let U be open and F a meager F_σ set with $A\Delta U \subseteq F$. Then, $G = U \setminus F$ is G_δ and $G \subseteq A$. Also, $M = A \setminus G \subseteq F$ is meager.

To show (1) $\implies$ (3) follows from (2) using $X \setminus A$. □

3.4 Baire Measurable Functions

Definition 3.7. *Let, X, Y be topological spaces and $f : X \to Y$ be a function, we say that f is* **Baire measurable** *if the inverse image of any open subset of Y has the Baire property in X.*

NOTE: If Y is second countable it is clearly enough to only consider the inverse images of a countable basis of Y.

For example, every continuous function is Baire measurable. If Y is metrizable then any function that is a pointwise limit of a sequence of continuous functions is Baire measurable.

Theorem 13. *Let X and Y be topological spaces and $f : X \to Y$ be Baire measurable. If Y is second countable there is a set $G \subseteq X$ that is a countable intersection of dense open sets such that $f\big|_G$ is continuous. In particular if X is Baire, f is continuous on a dense G_δ set.*

Proof. Let $\{U_n\}$ be a basis of Y. Then $f^{-1}(U_n)$ has a BP on X, so let V_n be open in X and let F_n be a countable union of closed nowhere dense sets with $f^{-1}(U_n)\Delta V_n \subseteq F_n$. Then $G = X \setminus F_n$ is a countable intersection of dense open sets and so $G = \bigcap_n G_n$. Since, $f^{-1}(U_n) \cap G = V_n \cap G$, $f\big|_G$ is continuous. □

3.5 Kuratowski-Ulam Theorem

We now consider sets in product spaces. We will see a Fubini-like theorem for Baire category.

Theorem 14. *Let X be a topological space, Y be a second countable space, $S \subseteq X \times Y$, $x \in X$ and $S_x \coloneqq \{y \in Y \mid (x, y) \in S\}$ be the vertical section of S at x.*

1. *If S is nowhere dense, then S_x is nowhere dense in Y for comeagerly many $x \in X$.*
2. *If S is meager, then S_x is meager in Y for comeagerly many $x \in X$.*

Proof.
1. We can assume that Y is not empty and S is closed. Let U be the complement of S. It is enough to show that U_x is dense for comeagerly many $x \in X$. Let $\{Y_n\}$ be a basis for the topology of Y made of nonempty sets. Then $U_n \coloneqq proj_X(U \cap (X \times Y_n))$ is dense open in X. If $x \in \bigcap_{n\in\mathbb{N}} U_n$, then $U_x \cap Y_n$ is not empty for all n, i.e., U_x is dense.
2. Follows from (1)

□

Theorem 15. *Let X, Y be countable spaces, $A \subseteq X$ and $B \subseteq Y$. Then $A \times B$ is meager if and only if A is meager or B is meager.*

Proof. If $A \times B$ is meager and A is not meager, then there is $x \in X$ such that $(A \times B)_x = B$ is meager, by Theorem 14.
Conversely, if A is meager and $A = \bigcup_{n\in\mathbb{N}} N_n$ with N_n nowhere dense, then $A \times B = \bigcup_{n\in\mathbb{N}} N_n \times B$, so it is enough to show that $N_n \times B$ is nowhere dense. This comes from the fact that if U is dense open in X, then $U \times Y$ is dense open in $X \times Y$. □

Theorem 16 (**Kuratowski-Ulam Theorem**)**.** *Let X, Y be second countable spaces, and $S \subseteq X \times Y$ having the BP.*

1. *S_x has the BP for comeagerly many $x \in X$. Similarly, $S^y := \{x \in X \mid (x, y) \in S\}$ has the BP for comeagerly many $y \in Y$*

2. *S is meager is equivalent to S_x is meager for comeagerly many $x \in X$, and to S^y is meager for comeagerly many $y \in Y$*

3. *S is comeager is equivalent to S_x is comeager for comeagerly many $x \in X$, and to S^y is comeager for comeagerly many $y \in Y$*

Proof. Let U be an open set and M be a meager set with $S \Delta U \subseteq M$.

1. For any $x \in X$, $S_x \Delta U_x \subseteq M_x$. By Theorem 14, S_x has the BP for comeagerly many $x \in X$.

2. By Theorem 14, if S is meager, then S_x is meager for comeagerly many $x \in X$. Conversely, if S is not meager, then U is not meager, which gives open sets $V \subseteq X$ and $W \subseteq Y$ such that $V \times W \subseteq U$ and $V \times W$ is not meager. By Theorem 15, V, W are not meager. This gives $x \in V$ such that S_x and M_x are meager. As $W \setminus M_x \subseteq U_x \setminus M_x \subseteq S_x, W \subseteq S_x \cup M_x$ is meager, a contradiction.

3. This comes from (2).

□

NOTE: Kuratowski-Ulam Theorem fails if S does not have the BP. For example, using the axiom of choice, there exists a non-meager set $S \subseteq [0, 1]^2$ so that no three points of S are in a straight line.