The Polyhedral Model

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Content

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3. Loop transformation: transforming loop by doing affine transformation on polytopes.
4. Dependency test: several mathematical methods for validating transformation on loop polytopes.
5. Code generation: generate transformed code from loop polytopes.

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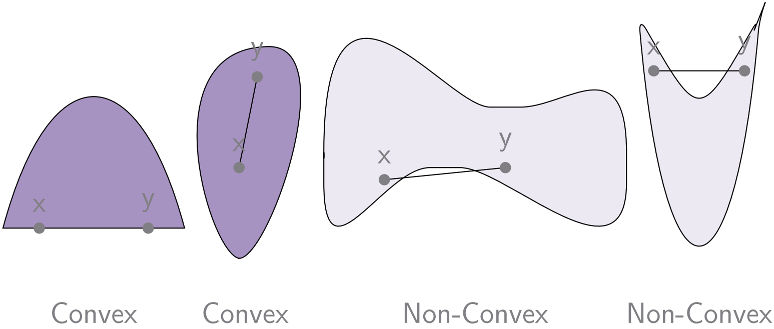
Definition

A set S is called convex if the line joining any two points in S is in S, i.e.,

$$\forall x, y \in S, \forall \lambda \in [0,1], \lambda x + (1-\lambda) y \in S.$$



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A $\mathbb{Z}$-polyhedron is a polyhedron where all its extreme points are integer valued.

In most situation loop counters are integers. So we use $\mathbb{Z}$-polyhedron to represent loop iteration domain.

Modeling Iteration Domains

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  for(j = 1; j <= n; j++){
      if (i <= n + 2 - j)
          b[j] = b[j] + a[i];
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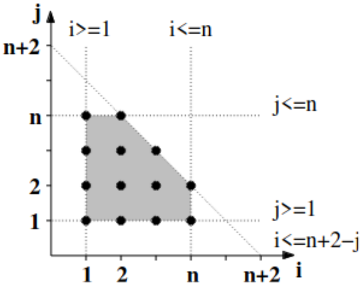

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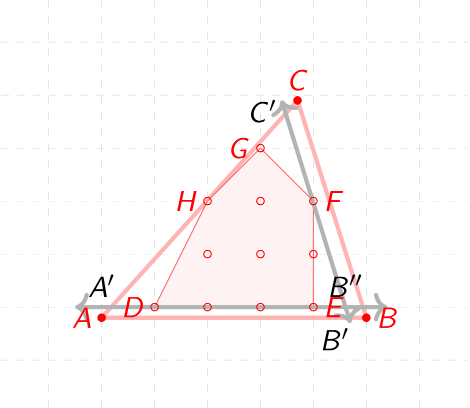
Definition

The integer hull P_I of a convex polyhedral set P is the convex hull of integer points of P.

Example



Definition

The dual representation of the polyhedron P can be expressed as a combination of lines L, rays R (forming the polyhedral cone) and vertices V (forming the polytope), that is
$\{x \in \mathbb{R}^n \mid \lambda L + \mu R + \rho V$ such that $\lambda, \mu, \rho \geq 0$ and $\lambda + \mu + \rho = 1\}$.

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Theorem (Fundamental theorem of polyhedral decomposition)

Every polyhedron P can be decomposed into a polytope V and a polyhedral cone L.

Definition

Given $\overrightarrow{m}$ the vector of symbolic parameters, a parametric polyhedron P is defined by $\{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} \leq B\overrightarrow{m} + \overrightarrow{b}\}$.

One-dimensional array

```
for(int i = 0; i < n; i++){
  for(int j = i + 1; j < n; j ++)
    A[i * n + j] = A[(n * j - n + j - i - 1];
    }
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1. Can we parallelize the two `for-loops`?
2. Is there data dependence between two different iterations of the nest?
3. Are there integer solutions to the following system of linear inequalities?

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 \times n + j_1 = n \times j_2 - n + j_2 - i_2 - 1 \end{cases}$$

Delinearize the array accesses

Linearized one-dimensional array

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Delinearized multi-dimensional array

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  for(int j = i + 1; j < n; j ++)
    B[i][j] = B[j - 1][j - i - 1];
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There is no integer solution, therefore, no dependence.

Bernstein Conditions

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Given two references, there exists a dependence between them if the following conditions are satisfied:

1. they reference the same array (cells)
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3. Output dependency (write-after-write), `B = 3, B = 7`

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4. Dependence polyhedron, list of sets of dependent instances

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Using this we can describe the dependence polyhedra of each dependence relation between two statements. It is a subset of cartesian product of iteration space R and S.

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2. Iteration domains: both S and R iteration domains can be described using affine inequalities: $A_S\overrightarrow{x_S} + c_S \geq 0$ and $A_R\overrightarrow{x_R} + c_R \geq 0$ respectively.

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3. Precedence order: each case corresponds to a common loop depth, and is called a dependence level.
   For each dependence level l, the precedence constraints are the equality of the loop index variables at depth lesser to l: $x_{R,i} = x_{S,i}$ for $i < l$ and $x_{R,l} > x_{S,l}$ if l is less than the common nesting loop level. Otherwise, there is no additional constraint and dependence exists if S is before R.
   Such constraints can be written using linear inequalities, $P_{l,S}\overrightarrow{x_S} - P_{l,R}\overrightarrow{x_R} + \overrightarrow{b} \geq 0$.

Algorithm 1 A Dependence Polyhedra Construction Algorithm

Require: Initialize reduced dependence graph with one node per statement
Ensure: Dependence polyhedra

```
for all pair $R$, $S$ do
    for all distinct references $f_R$, $f_S$ to the same array do
        if commonLoops($R, S$) $= \emptyset$ then
            minDepth = 0
        else
            minDepth = 1
        end if
        for $l$ = minDepth to |commonLoops| do
            Build $\mathcal{D}_{R,S}$
            if $\mathcal{D}_{R,S} \neq \emptyset$ then
                type = concatenate(type($f_R$), A, type($f_S$)) { WAW, RAW, WAR, RAR}
            end if
            addDegree($R, S, \{l, \mathcal{D}_{R,S}, type\}$)
        end for
    end for
end for
```

Example of Dependence Polyhedron

```
for (i = 0; i <= n; i++){
    for (j = 0; j <= n; j++)
        a[i][j] = a[i+1][j+1]; //S1 }
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Iteration Domain:

$$\mathcal{D}_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix} \geq \vec{0}$$

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Array Reference Function:

$$F_{A}\overrightarrow{x_{S1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix} \qquad F'_{A}\overrightarrow{x_{S1}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix}$$

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Precedence Order:

For statement `S1` in two consecutive loop, `i - i' = 1, j - j' = 1,`

$$P_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix}$$

To satisfy, $P_{S1}\overrightarrow{x_{S1}} - P_R\overrightarrow{x_R} + \overrightarrow{b} \geq \overrightarrow{0}$, where $\overrightarrow{b} \in [-1, 1]$.

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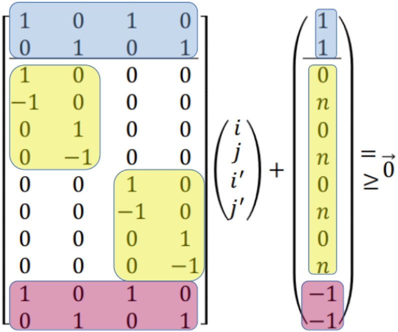

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{pmatrix} i \\ j \\ i' \\ j' \end{pmatrix} + \left(\begin{array}{c} 1 \\ 1 \\ \hline 0 \\ n \\ 0 \\ n \\ 0 \\ n \\ 0 \\ n \\ -1 \\ -1 \end{array}\right) \begin{smallmatrix} = \\ \geq \end{smallmatrix} \vec{0}$$

🟦 Array Reference Function

🟨 Iteration Domain

🟥 Precedence Order

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Definition

Given a statement S, a p-dimensional affine schedule Θ_S is an affine form on the outer loop iterators $\overrightarrow{x_S}$ and the global parameters $\overrightarrow{n}$,

$$\Theta_S(\overrightarrow{x_S}) = T_S \begin{pmatrix} \overrightarrow{x_S} \\ \overrightarrow{n} \\ 1 \end{pmatrix}$$

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where $T_S \in \mathbb{R}^{p \times \dim(\overrightarrow{x_S}) + \dim(\overrightarrow{n}) + 1}$.

- If T_S is a vector, Θ_S is called a one-dimensional schedule.
- If T_S is a matrix, Θ_S is called a multi-dimensional schedule.

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4. A multidimensional schedule expresses the program as one or more nested sequential loops, possibly enclosing one or more parallel loops.
5. Program transformation in the polyhedral model can be specified by a well chosen scheduling function.

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6. Dependence graph can be used to represent scheduling constraints between the program operations.
7. Hyperplanes can be interpreted as schedules.

Example of one-dimensional schedule

Time	Code	Time Stamp
`T = 0`	`x = a + b; //S1`	`T_S1 = 0;`
`T = 1`	`y = a + b; //S2`	`T_S2 = 1;`
`T = 2`	`z = x + y; //S3`	`T_S3 = 2;`

Function `T` returns the logical date of each statement.

Example of multi-dimensional schedule

```
 Time              Code                         Time Stamp
T = 0         x = a + b; //S1                   T_S1 = (0);
T = 1         for (i = 0; i < 2; i ++){
  i = 0             a[i] = x; //S2              T_S2(0) = (1,0);
  i = 1             }                           T_S2(1) = (1,1);
T = 2         z = x + y; //S3                   T_S3 = (2);
```

Function `T` returns the logical date of each statement.
Logical dates may be multi-dimensional:

- Lexicographical Order: $T_{S1} < T_{S2} < T_{S3} \iff (0) < (1, i) < (2)$.

Unlike one-dimensional schedules, it is always possible to build a legal multidimensional schedule for a SCoP.

Theorem ([Fea97])

Every static control program has a multi-dimensional affine schedule.

1. Bernstein conditions are useful to decide if a program transformation is legal if $\begin{cases} \mathcal{W}_a \cap \mathcal{W}_b = \emptyset \\ \mathcal{W}_a \cap \mathcal{R}_b = \emptyset \\ \mathcal{R}_a \cap \mathcal{W}_b = \emptyset \end{cases}$.

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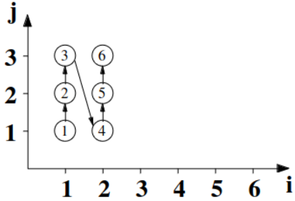
Definition (Precedence condition)

Θ_R and Θ_S are legal schedule for instances R and S respectively if for all $< \overrightarrow{x_R}, \overrightarrow{x_S} > \in \mathcal{D}_{R,S}$ then $\Theta_R(\overrightarrow{x_R}) < \Theta_S(\overrightarrow{x_S})$ holds.

Example of Transformation (Loop Interchange)

Original

```
for(i = 1; i <= 2; i++){
  for(j = 1; j <= 3; j++)
      b[i][j] = ...; } //S1
```



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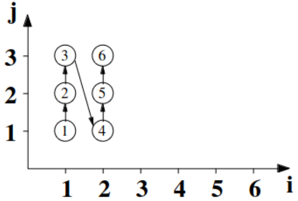

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \overrightarrow{0}$$

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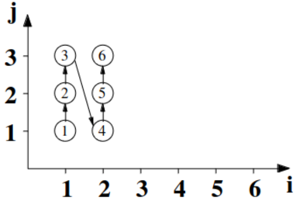

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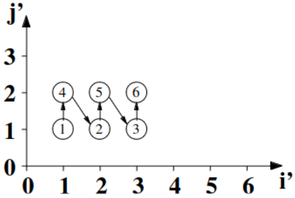

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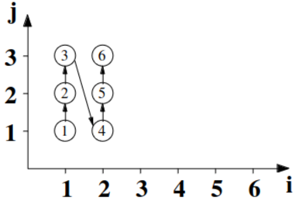

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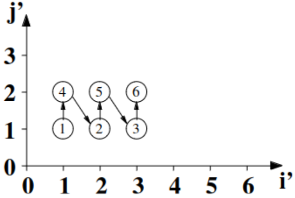

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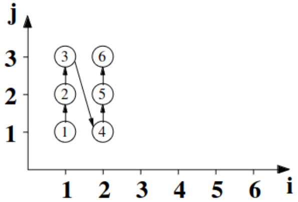

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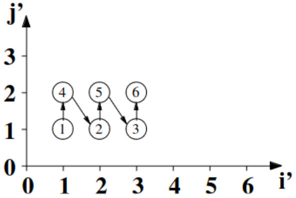

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Transformation Function: $\begin{pmatrix} i' \\ j' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} i \\ j \end{pmatrix} \geq \vec{0}$.

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T_S1(i,j) = (i ,j);
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Example of Transformation (Loop Reversal)

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```
for(i = 1; i <= 2; i++){
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      b[i][j] = ...; } //S1
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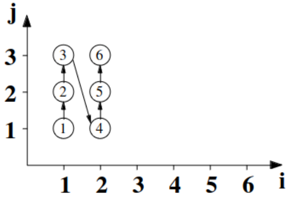

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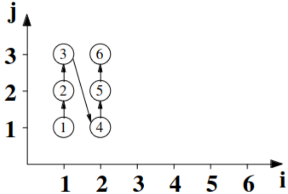

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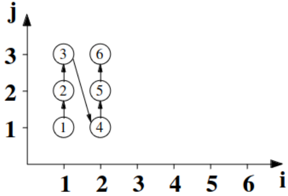

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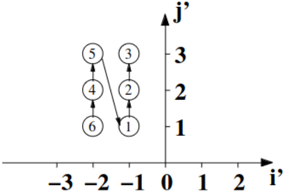

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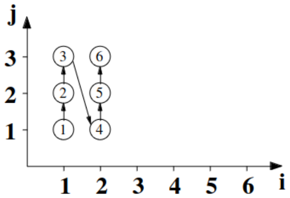

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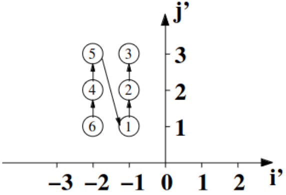

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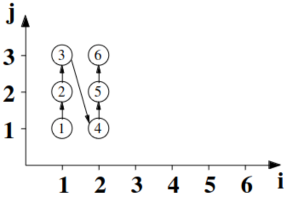

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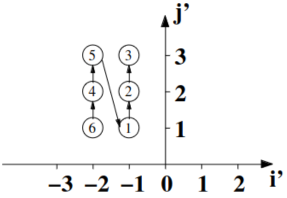

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Loop Tiling

Loop tiling [IT88, WL91, Xue00] is a key transformation in optimizing for parallelism and data locality.

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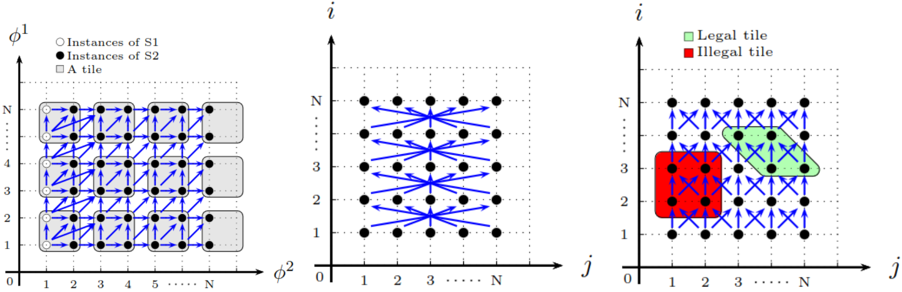
Theorem

Two one-dimensional schedules are valid tiling hyperplanes if and only if they satisfy the precedence conditions.



Example of Tiling: Transpose matrix vector multiply [BRS10]

Original code

```
for(i = 0; i < N; i++){
  P: x[i] = 0;
  for(j = 0; j < N; j++)
      Q: x[i] += a[j][i] * y[j];
  }
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Original code

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for(i = 0; i < N; i++){
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Iteration Space

$$\mathcal{D}_Q^{\text{orig}} \begin{pmatrix} i \\ j \\ N \\ 1 \end{pmatrix} \geq 0, \ \mathcal{D}_Q^{\text{tiled}} \begin{pmatrix} it \\ jt \\ i \\ j \\ N \\ 1 \end{pmatrix} \geq 0$$

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Tiled code

```
for(it = 0; it <= floord(N - 1, 32); it++){
  for(jt = 0; jt <= floord(N - 1, 32); jt++)
    if(jt == 0){
      for(i = max(32 * it, 0);
        i <= min(32 * it + 31, N - 1); i++){
          P: x[i] = 0;
          Q: x[i] = x[i] + a[0][i] * y[0];
          }
      }
    for(i = max(32 * it, 0);
        i <= min(32 * it + 31, N - 1); i++){
      for(j = max(32 * jt, 1);
        j <= min(32 * jt + 31, N - 1); j++){
        Q: x[i] = x[i] + a[j][i] * y[j];
      }
    }
  }
```

Drawbacks of the Polyhedral Model

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2. Building polyhedrons in compile time is also memory consuming.

References I

- Louis-Noël Pouchet.
  Polyhedral Compilation Foundations, 2010.
  https://www.cs.colostate.edu/~pouchet/lectures/888.11.lect1.html#lect1
- Cédric Bastoul.
  Improving Data Locality in Static Control Programs.
  Ph.D. Thesis, 2004.
- Fangzhou Jiao.
  An Overview to Polyhedral Model, 2010.
  https://homes.luddy.indiana.edu/achauhan/Teaching/B629/2010-Fall/StudentPresns/PolyhedralModelOverview.pdf

References II

- Paul Feautrier.
  Some efficient solutions to the affine scheduling problem: Part II Multidimensional time
  nternational Journal of Parallel Programming, 1997.
- Francois Irigoin and Rémi Triolet.
  Supernode partitioning.
  ACM SIGPLAN Principles of Programming Languages, 1988.
- Michael E. Wolf and Monica S. Lam.
  A data locality optimizing algorithm.
  ACM SIGPLAN symposium on Programming Languages Design and Implementation, 1991.

References III

Jingling Xue.
Loop tiling for parallelism.
Kluwer Academic Publishers, 2000.

Muthu Manikandan Baskaran, Jeyakumar Ramanujam and Ponnuswamy Sadayappan.
Automatic C-to-CUDA Code Generation for Affine Programs.
International Conference on Compiler Construction, 2010.