Using Saturation Matrix to Efficiently Remove Redundant Inequalities

Chirantan Mukherjee

University of Western Ontario

June 25, 2024

Content

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$$\begin{array}{rcl} -2\,x_1 + 4\,x_2 - 3\,x_3 = 0 & & -2\,x_1 + 4\,x_2 - 3\,x_3 = 0 \\ -13\,x_1 + 24\,x_2 - 20\,x_3 = 0 & \xrightarrow{1 \; step \;\; GE} & 0\,x_1 - 2\,x_2 - \frac{1}{2}\,x_3 = 0 \\ -26\,x_1 + 54\,x_2 - 39\,x_3 = 0 & & 0\,x_1 + 2\,x_2 - 0\,x_3 = 0 \end{array}$$

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$$\begin{aligned} 3\,x_1 - 2\,x_2 + 1\,x_3 &\le 7 \\ -2\,x_1 + 2\,x_2 - 1\,x_3 &\le 12 \\ -4\,x_1 + 1\,x_2 - 3\,x_3 &\le 15 \end{aligned} \xrightarrow{1\ \ step\ \ FME} \begin{aligned} 0\,x_1 + 2\,x_2 - 1\,x_3 &\le 50 \\ 0\,x_1 - 5\,x_2 - 13\,x_3 &\le 73 \end{aligned}$$

Example

Eliminating t_1 from

$$A = \begin{cases} a_1 : 3t_1 - 2t_2 + t_3 \leq 7 \\ a_2 : -2t_1 + 2t_2 - t_3 \leq 12 \\ a_3 : -4t_1 + t_2 + 3t_3 \leq 15 \end{cases}$$

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$combine(a_1, a_3) = combine(4a_1 + 3a_3) = -5t_2 - 13t_3 \leq 73$

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$$A' = \begin{cases} a_4 : 2t_2 - t_3 \leq 50 \\ a_5 : -5t_2 - 13t_3 \leq 73 \end{cases}$$

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$combine(a_1, a_2) = combine(5a_4 + 2a_5) = -31t_3 \leq 396$

$$A'' = \begin{cases} -31t_3 \leq 396 \end{cases}$$

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- Projection of polyhedral sets has many applications in computer science
  - scheduling
  - dependence analysis (automatic parallelization)

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- In [JMXY24] proposed a method using Saturation Matrix.

Definition (H-representation)

A polyhedron P is a set which can be expressed as the intersection of finite number of (closed) half spaces, that is $\{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} \leq \overrightarrow{b}\}$.

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Definition (V-representation)

The dual representation of the polyhedron P can be expressed as a combination of rays R (forming the polyhedral cone) and vertices V (forming the polytope), that is
$\{x \in \mathbb{R}^n \mid \mu R + \rho V$ such that $\mu, \rho \geq 0$ and $\mu + \rho = 1\}$.

We will denote it as $\mathcal{VR}(F)$, where F is the H-representation of P.

Definition

Having two inequality: $a_1x_1 + \cdots + a_nx_n \leq d_1$ and $b_1x_1 + \cdots + b_nx_n \leq d_2$ such that $a_1 > 0$ and $b_1 < 0$, we can eliminate x_1 by multiplying the first inequality by $|b_1|$ and the second one by a_1 and add them together. The result of combining these inequalities is:

$$(a_2|b_1| + b_2a_1)x_2 + \cdots + (a_n|b_1| + b_na_1)x_n \leq |b_1|d_1 + a_1d_2.$$

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Definition

Having a linear inequality system S with m inequalities and n variables of the form $a_{i1}x_1 + \cdots + a_{in}x_n \leq d_i$, We can partition the inequalities in three groups with respect to x_1:

- A^+ set of inequalities with positive x_1 coefficient.
- A^- set of inequalities with negative x_1 coefficient.
- A^0 set of inequalities with zero x_1 coefficient.

Idea

Theorem

Let A' be the union of combination of all inequalities in A^+ with all inequalities in A^- and inequalities in A^0 such that A' does not have x_1. Then,

$$(x_2, \cdots, x_n) \in Sol(A') \Longleftrightarrow \exists x_1 \; (x_1, x_2, \cdots, x_n) \in Sol(A)$$

where $Sol(A)$ is a set of real points which satisfies all inequalities in A.

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- Select variables one after another.
- Partition A^+, A^- and A^0 with respect to the variable.
- Combine inequalities in A^+ and A^- and form the resulting union.

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Improve Complexity

FME's complexity is double exponential. Most of the inequalities generated by FME algorithm are redundant. Detecting these redundant inequalities and removing them can significantly improve algorithm's complexity.

Definition

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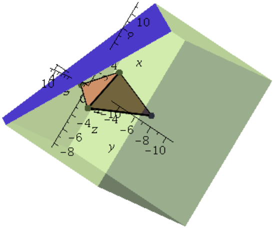
- Weakly redundant if it is redundant and $\overrightarrow{a}^t\overrightarrow{x} = \overrightarrow{b}$ holds for some $\overrightarrow{x} \in P$.



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  - It is not practical for large cases.
- It neither has a good theoretical complexity, nor is it effective in practice because of its dependence on LP solvers.
- Since, FME are essentially an adaptation of GE, it is desirable to achieve the redundancy via linear algebra instead of using LP.

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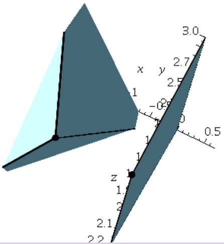
Example

Consider the polyhedron defined by $\{x + 2\,y - z \leq 2, 2\,x - 3\,y + 6\,z \leq 2, -2\,x + 3\,y + 4\,z \leq 20\}$. Its projection on $[y, z]$ is the polyhedron represented by $\{z \leq \frac{11}{5},\ y + \frac{2}{7}z \leq \frac{24}{7}\}$.



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  - First construct the initial test cone from the input polyhedron.
  - This cone can be used to find the "polar cone" of the polyhedron after projection.
  - Redundant inequalities can be detected using extreme rays of the polar cone.
- For a non-empty, full-dimensional, and pointed polyhedron $P \subset \mathbb{Q}^n$ as input, given by a system of m linear inequalities of height h, the complexity is $\mathcal{O}(m^{\frac{5n}{2}} n^{\omega+1+\epsilon} h^{1+\epsilon})$ bit operations, for any $\epsilon > 0$, where ω is the exponent of matrix multiplication.

New Algorithm in Maple

Saturation Matrix

Definition

The saturation matrix of a system of linear inequalities F is the Boolean matrix $\mathrm{satM}(F) \in \mathbb{Q}^{m \times k}$, whose (i, j)-th element is equal to 1, if the j-th element of $\mathcal{VR}(F)$ saturates the i-th inequality of F, 0 otherwise.

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Example

Consider the system F with the set $\mathcal{VR}(F)$ and the saturation matrix $\text{satM}(F)$ given below.

F	$\mathcal{VR}(F)$
$\ell_1 : x + y \leq 1$	$\vec{v}_1 : (0, 1)$
$\ell_2 : -x - y \leq 1$	$\vec{v}_2 : (1, 0)$
$\ell_3 : x - y \leq 1$	$\vec{v}_3 : (-1, 0)$
$\ell_4 : -x + y \leq 1$	$\vec{v}_4 : (0, -1)$

$\text{satM}(F)$

	$\vec{v}_1$	$\vec{v}_2$	$\vec{v}_3$	$\vec{v}_4$
ℓ_1	1	1	0	0
ℓ_2	0	0	1	1
ℓ_3	0	1	0	1
ℓ_4	1	0	1	0

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Observation:

The composition of the above two can be denotes by $\mathcal{S}^{\mathcal{H}}(\mathcal{S}^{\mathcal{VR}}(\ell))$, which is the collection of inequalities saturated by all the vertices or rays saturating ℓ.

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3. *The inequality ℓ is weakly redundant in F if and only if the set $\mathcal{S}^{\mathcal{H}}(\mathcal{S}^{\mathcal{VR}}(\ell)) \setminus \{\ell\}$ is not empty.*

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2. *If $\mathcal{S}^{\mathcal{VR}}(\ell)$ is non-empty and its cardinality is less than n, then the inequality ℓ is weakly redundant in F.*
3. *The inequality ℓ is weakly redundant in F if and only if the set $\mathcal{S}^{\mathcal{H}}(\mathcal{S}^{\mathcal{VR}}(\ell)) \setminus \{\ell\}$ is not empty.*

Corollary (Criteria using the saturation matrix)

1. *If* $\mathrm{satM}(F)[\ell]$ *contains zero elements, then ℓ is strongly redundant.*

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2. *If the number of nonzero elements of $\mathrm{satM}(F)[\ell]$ is positive and less than the dimension n, then ℓ is weakly redundant.*

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3. *If $\mathrm{satM}(F)[\ell]$ is contained in $\mathrm{satM}(F)[\ell_1]$ for some $\ell_1 \in F \setminus \{\ell\}$, then ℓ is weakly redundant.*

From the saturation matrix $\mathrm{satM}(F) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, it is easy to obtain the following identities:

$$\begin{aligned} \mathcal{S}^{\mathcal{VR}}(\ell_1) &= \{\overrightarrow{v}_1, \overrightarrow{v}_2\} & \mathcal{S}^{\mathcal{H}}(\overrightarrow{v}_1) &= \{\ell_1, \ell_4\} \\ \mathcal{S}^{\mathcal{VR}}(\ell_2) &= \{\overrightarrow{v}_3, \overrightarrow{v}_4\} & \mathcal{S}^{\mathcal{H}}(\overrightarrow{v}_2) &= \{\ell_1, \ell_3\} \\ \mathcal{S}^{\mathcal{VR}}(\ell_3) &= \{\overrightarrow{v}_2, \overrightarrow{v}_4\} & \mathcal{S}^{\mathcal{H}}(\overrightarrow{v}_3) &= \{\ell_2, \ell_4\} \\ \mathcal{S}^{\mathcal{VR}}(\ell_4) &= \{\overrightarrow{v}_1, \overrightarrow{v}_3\} & \mathcal{S}^{\mathcal{H}}(\overrightarrow{v}_4) &= \{\ell_2, \ell_3\} \end{aligned}$$

Algorithm 1 Check Redundancy

Require: 1. the inequality system F with m inequalities;
2. the saturation matrix satM(F).

Ensure: the minimal irredundant system F_{ired}.

$F_{\text{irred}} := \{ \}$ and $\text{satM}_{\text{irred}} := [\ ]$.
for i from 1 to m **do**
  Let *Redundant* := *False*.
  if the number of nonzero elements in satM[i] is less than n **then**
    next. /* Corollary 1 and 2*/
  end if
  for j from 1 to $i - 1$ **do**
    if satM[i] = satM[i]&satM[j] **then**
      Redundant := True.
      break. /* Corollary 3 */
    end if
  end for
  if not Redundant **then**
    $F_{\text{irred}} := F_{\text{irred}} \cup \{f_i\}$ and append satM[i] to $\text{satM}_{\text{irred}}$.
  end if
end for

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- For any vertices $\{v_1, \ldots, v_e\}$ so that $\mathrm{proj}(v_1, \{x\}) = \cdots = \mathrm{proj}(v_e, \{x\})$ holds, we merge in $\mathrm{satM}(F)$ the saturation information contained by these columns indexed by $v_1, \ldots, v_e$ as follows:

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    - we compute the bit-wise OR of the columns (regarded as bit-vectors) of $\mathrm{satM}(F)$ indexed by $v_1, \ldots, v_e$
    - we replace the columns indexed by $v_1, \ldots, v_e$ by this bit-wise OR.

Example

- $\mathcal{S}^{\mathcal{VR}}(\ell_1) = \{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ and $\mathcal{S}^{\mathcal{VR}}(\ell_4) = \{\overrightarrow{v}_1, \overrightarrow{v}_3\}$.

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- $\text{Merge}(\text{satM}(F)[\ell_1] \& \text{satM}(F)[\ell_4]) = (1, 0, 0)$.
- $\text{proj}(\{\ell_1, \ell_4\}, \{x\}) = \{x \leq 1\}$.

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- $\mathrm{Merge}(\mathrm{satM}(F)[\ell_1] \& \mathrm{satM}(F)[\ell_4]) = (1, 0, 0)$.
- $\mathrm{proj}(\{\ell_1, \ell_4\}, \{x\}) = \{x \leq 1\}$.
- Among the four vertices, only $\mathrm{proj}(\overrightarrow{v}_1, \{x\})$ saturates $\mathrm{proj}(\{\ell_1, \ell_4\}, \{x\})$.

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With the techniques of updating the saturation matrix, we provide Algorithm to compute the minimal projected representation of a polyhedron.

Algorithm 2 Minimal projected representation

Require: 1. an inequality system F;
2. a variable order $x_1 > x_2 > \ldots > x_d$.

Ensure: the minimal projected representation *res* of F.

Compute the V-representation V of F;
Set $res :=$ table().
Sort the elements in V w.r.t. the reverse lexico order.
$F := \mathrm{CheckRedundancy}(F)$.
$res[x_1] := F^{x_1}$.
for i from 1 to $n-1$ **do**
  $(F^p, F^n, F^0) :=$ partition(F).
  Let $V_{new} :=$ proj$(V, \{x_i\})$.
  Let $F_{new} := F^0$.
  for each $f_p \in F^p$ and $f_n \in F^n$ **do**
    Append proj$((f_p, f_n), \{x_i\})$ to F_{new}
  end for
  $F := \mathrm{CheckRedundancy}(F_{new})$.
  $V := V_{new}$, $res[x_{i+1}] := F^{x_{i+1}}$.
end for
return *res*.

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Set $res := \text{table}()$.
Sort the elements in V w.r.t. the reverse lexico order.
Compute the saturation matrix satM**.**
F, satM $:= \mathrm{CheckRedundancy}(F,$ satM(F)$)$.
$res[x_1] := F^{x_1}$.
for i from 1 to $n-1$ **do**
    $(F^p, F^n, F^0) := \text{partition}(F)$.
    Let $V_{new} := \text{proj}(V, \{x_i\})$.
    Merging: satM := Merge(satM)**.**
    Let $F_{new} := F^0$ **and** $\text{satM}_{new} := \text{satM}[F^0]$.
    for each $f_p \in F^p$ and $f_n \in F^n$ **do**
        Append $\text{proj}((f_p, f_n), \{x_i\})$ to F_{new}
        Append $\text{satM}[f_p]\&\text{satM}[f_n]$ **to** satM_{new}.
    end for
    F, satM $:= \mathrm{CheckRedundancy}(F_{new}, \text{satM}_{\mathbf{new}})$.
    $V := V_{new}$, $res[x_{i+1}] := F^{x_{i+1}}$.
end for
return *res*.

Complexity

Input H-representation $(A, \overrightarrow{b})$ with $A \in \mathbb{Q}^{m \times n}$, $\overrightarrow{b} \in \mathbb{Q}^m$ and $\text{height}([A, \overrightarrow{b}]) = h$.

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  - For a fixed inequality ℓ in F, find the index set I of all the 1's in satM$[\ell]$.

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  - For a fixed inequality ℓ in F, find the index set I of all the 1's in satM$[\ell]$.
  - Then, apply bit-wise AND to column vectors of satM$[1..-1, I]$. This requires $m \cdot |I|$ bit operations, where $|I| < k$ is the cardinality of I.

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- Redundancy detection in the initial input system: –> $\mathcal{O}(m^{n+2})$ bit operations.
  - For a fixed inequality ℓ in F, find the index set I of all the 1's in $\text{satM}[\ell]$.
  - Then, apply bit-wise AND to column vectors of $\text{satM}[1..-1, I]$. This requires $m \cdot |I|$ bit operations, where $|I| < k$ is the cardinality of I.
  - Redundancy detection for one inequality requires at most m^{n+1} bit operations. Therefore, the the redundancy detection for the input system F requires at most m^{n+2} bit operations.

Comparision of Algorithms

For a non-empty, full-dimensional, and pointed polyhedron $P \subset \mathbb{Q}^n$ as input, given by a system of m linear inequalities of height h, the complexity of eliminating d $(\leq n)$ variables, where $\epsilon > 0$, ω denotes the exponent of matrix multiplication and $LP(d, H)$ is an upper bound for the number of bit operations required for solving a linear program in n variables and with total bit size H.

FME Algorithms	Complexity
Original	$\mathcal{O}(m^{2^d})$
Linear Programming	$\mathcal{O}(d^2 m^{2d} LP(d, 2^d h d^2 m^d))$
Balas and Kohler Check	$\mathcal{O}(m^{\frac{5n}{2}} n^{\omega+1+\epsilon} h^{1+\epsilon})$
Saturation Matrix	$\mathcal{O}(m^{2n} n^{\omega+\epsilon} h^{1+\epsilon})$

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