III. REGULAR EXPRESSIONS

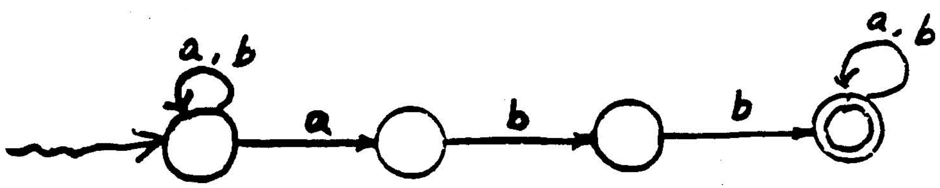
— The Second Model for Defining Languages

Example:

Consider the language of all words that consist of a's and b's and have abb as a subword. We can formally define this language in the following ways:

(1) $L = \{x \mid x \in \{a, b\}^*$ **and** x **has** abb **as a subword**$\}$;

(2) $L = L(M)$ **where** M **is an NFA given by the following diagram:**



Both of the above definitions are lengthy. It can also be expressed by

$$L((a+b)^*abb(a+b)^*)$$

Definition Let Σ be an alphabet.

A regular expression over Σ is defined recursively:

Basis: **(1)** $\emptyset$,

(2) ε,

(3) a, **where** $a \in \Sigma$

are R.E.'s over Σ

Induction Step:

If E_1 **and** E_2 **are R.E.'s over** Σ, **then**

(4) $(E_1 + E_2)$,

(5) $(E_1 \cdot E_2)$,

(6) E_1^*

are R.E.'s over Σ

We usually omit the operator '$\cdot$'.

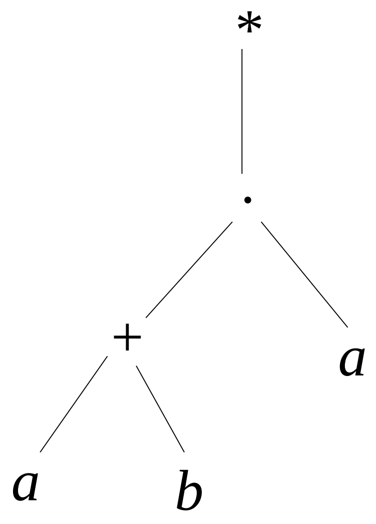
The set of all regular expressions over Σ is denoted by

$$\mathcal{R}_\Sigma$$

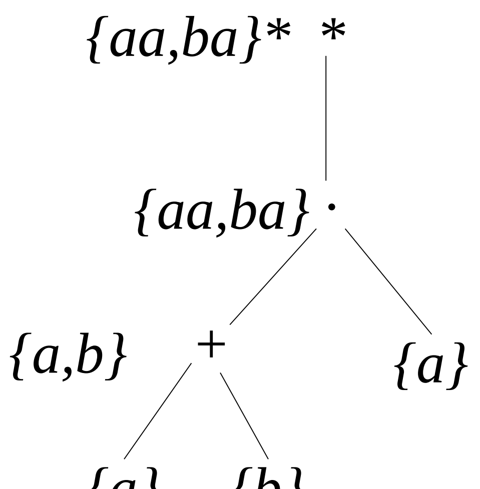
Example:

$$(\underbrace{\underbrace{(\overbrace{a}^{E_1}+\overbrace{b}^{E_2})}_{E_3}\underbrace{a}_{E_4}}_{E_5})^* \textbf{ is a R.E. over } \{a,\ b\}$$

We display the "parsing" by an expression tree:



What language does an R.E. define?



Definition Given a regular expression E, the language $L(E)$ denoted by E is defined as follows:

Basis: (1) if $E = \emptyset$, then $\emptyset$;
(2) if $E = \varepsilon$, then $\{\varepsilon\}$;
(3) if $E = a$, then $\{a\}$;

Induction Step:
(4) if $E = (E_1 + E_2)$, then $L(E_1) \cup L(E_2)$;
(5) if $E = (E_1 E_2)$, then $L(E_1)L(E_2)$;
(6) if $E = E_1^*$, then $(L(E_1))^*$.

Properties of R.E.'s

$+$: $E_1 + E_2 \equiv E_2 + E_1$;
$(E_1 + E_2) + E_3 \equiv E_1 + (E_2 + E_3)$;
So, $(E_1 + E_2) + E_3 \Rightarrow E_1 + E_2 + E_3$.

$\cdot$: $(E_1 E_2)E_3 \equiv E_1(E_2 E_3)$;
So, $(E_1 E_2)E_3 \Rightarrow E_1 E_2 E_3$.

. . . .

Definition **A language L is regular iff there is a regular expression E such that $L = L(E)$.**

The family of (all) regular languages is denoted by $\mathcal{L}_{REG}$.

Example $E = (b^*ab^*a)^*b^*$.

$L_{even} = \{x \mid x$ **in** $\{a,b\}^*$ **and** x **contains an even number of** $a's\}$.

Claim: $L(E) = L_{even}$

Proof:

(i) $L(E) \subseteq L_{even}$ **since every word in** $L(E)$ **contains an even number of** a**'s.**

(ii) Let $x \in L_{even}$. **Then** x **can be written as** $b^{i_0}ab^{i_1}a \ldots\ldots ab^{i_{2n}},\ i_0, i_1, \ldots, i_{2n} \geq 0$.

So, $x = (b^{i_0}ab^{i_1}a)(b^{i_2}ab^{i_3}a) \ldots\ldots (b^{i_{2n-2}}ab^{i_{2n-1}}a)b^{i_{2n}}$.

$x \in L((b^*ab^*a)^*)L(b^*) \ = \ L(E)$. $\qquad$ *q.e.d.*

<u>**Examples**</u> $\Sigma = \{a,\ b\}$.

$$L_1 = \{x \mid x = au,\ u \in \Sigma^*\}.$$

$$L_2 = \{x \mid |x|_a \equiv 0 \ \textbf{mod}\ 3\}.$$

$$L_3 = \{x \mid x \textbf{ has 2 or 3 } a's \textbf{ with the last two appearances nonconsecutive } \}$$

$$L_4 = \{x \mid x = a^n b^n,\ n \geq 1\}$$

Examples

What are the languages denoted by the following R.E.'s ?

$$E_1 = a^*ba^*$$

$$E_2 = (a + ab)^*$$

$$E_3 = a(a + b)^*a$$

$$E_4 = (aa + bb + ba + ab)^*$$

How many languages over Σ do R.E.'s define?

(1) Infinitely many ?

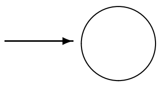
(2) Countable ?

Regular Expressions into Finite Automata

Let E **be a regular expression over** Σ**. Then we can construct a** ε**-NFA** M **such that** $L(M) = L(E)$**, using the following rules:**

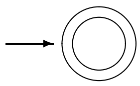
(i) $E = \emptyset$**.**

Construct M **such that** $L(M) = \emptyset$



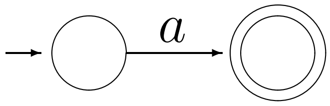
(ii) $E = \varepsilon$**.**

Construct M **such that** $L(M) = \{\varepsilon\}$



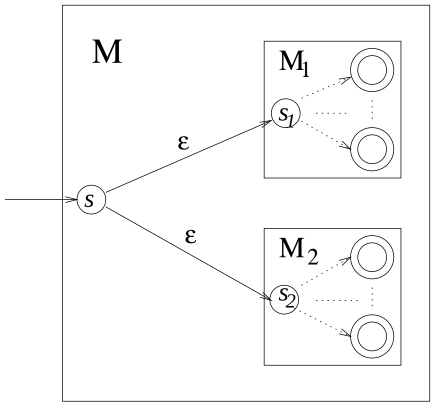
(iii) $E = a$**,** $a \in \Sigma$**.**

Construct M **such that** $L(M) = \{a\}$



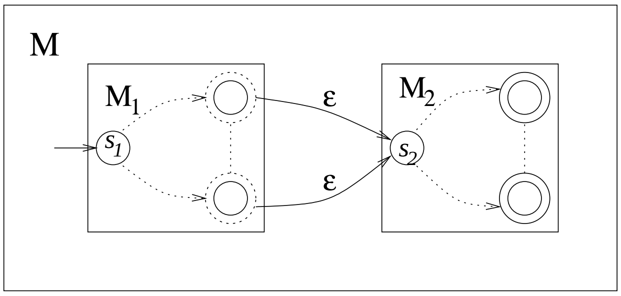
(iv) $E = (E_1 + E_2)$**.**

Construct M **such that** $L(M) = L(M_1) \cup L(M_2)$**.**



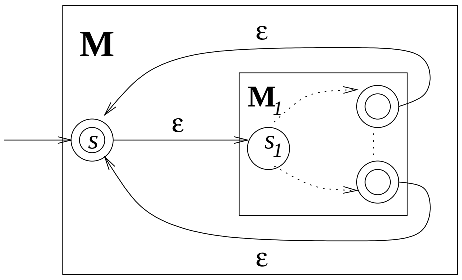
(v) $E = (E_1 E_2)$**.**

Construct M **such that** $L(M) = L(M_1)L(M_2)$**.**



(vi)$E = E_1^*$**.**

Construct M **such that** $L(M) = L(M_1)^*$**.**



Example

$E = (c^*(a + (bc^*))^*)$

Construct a FA M such that $L(M) = L(E)$.

$\triangle$ $a, \quad b, \quad c$ **by (iii)**

$\triangle$ c^* **by (vi)**

$\triangle$ (bc^*) **by (v)**

$\triangle$ $(a + bc^*)$ **by (iv)**

$\triangle$ $(a + bc^*)^*$ **by (vi)**

$\triangle$ $(c^*(a + bc^*)^*)$ **by (v)**

Theorem For E, an arbitrary regular expression over Σ, the ε-NFA, M, constructed as above satisfies $L(M) = L(E)$.

Proof: Let $Op(E)$ be the total number of $+, \cdot$, and * operations in E. We prove this theorem by induction on $Op(E)$.

Basis: $Op(E) = 0$. Then $E = \emptyset,\ \varepsilon,$ or $a \in \Sigma$. Then clearly we have $L(M) = L(E)$.

Induction Hypothesis:
Assume that the claim holds for all E with $Op(E) \leq k$, for some $k \geq 0$.

Induction Step:
Consider an arbitrary regular expression E with $Op(E) = k+1$. Since $k+1 \geq 1,\ E$ contains at least one operator $+,\ \cdot$, or *.

Case I: $E = E_1 + E_2$. **Then** $Op(E_1) \leq k$ **and** $Op(E_2) \leq k$. **So,** $L(M_1) = L(E_1)$ **and** $L(M_2) = L(E_2)$ **by I.H.. We know the construction of** $M = M_1 \cup M_2$ **satisfies** $L(M) = L(M_1) \cup L(M_2)$**, and** $L(E) = L(E_1) \cup L(E_2)$**. Therefore,** $L(M) = L(E)$**.**

Case II: $E = E_1 E_2$

. . . . .

Case III: $E = E_1^*$

. . . . .

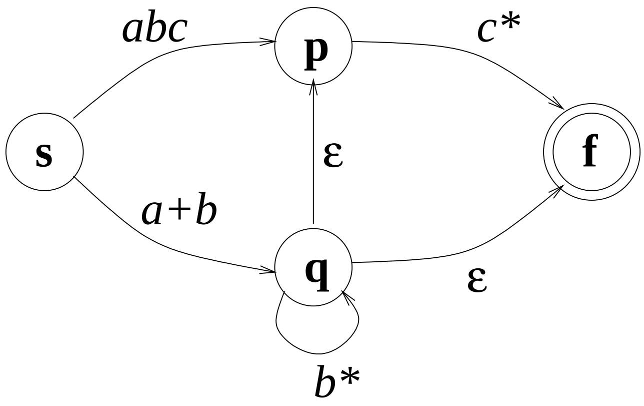
In each of the three cases, we have shown that $L(M) = L(E)$**. Therefore this holds for all regular expressions by the principle of induction.** *q.e.d.*

Finite Automata into Regular Expressions

To prove that every DFA language is regular we introduce an extension of finite automata.

Definition An extended finite automaton (EFA), M, is a quintuple $(Q,\ \Sigma,\ \delta,\ s,\ f)$ where
$Q,\ \Sigma,\ s$ are as in ε-NFA,
f is the only final state, $f \neq s$,
$\delta : Q \times Q \rightarrow R_\Sigma$ is a total extended transition function.

Example of an EFA:



$\delta(p, s) = \emptyset$, $\delta(s, f) = \emptyset$, ...
One final state $f \neq s$

△ **A <u>configuration</u> is in** $Q\Sigma^*$

△ **<u>Move</u>**

$$px \;\vdash\; qy \textbf{ if}$$

(i) $x = wy,\ w \in \Sigma^*$,

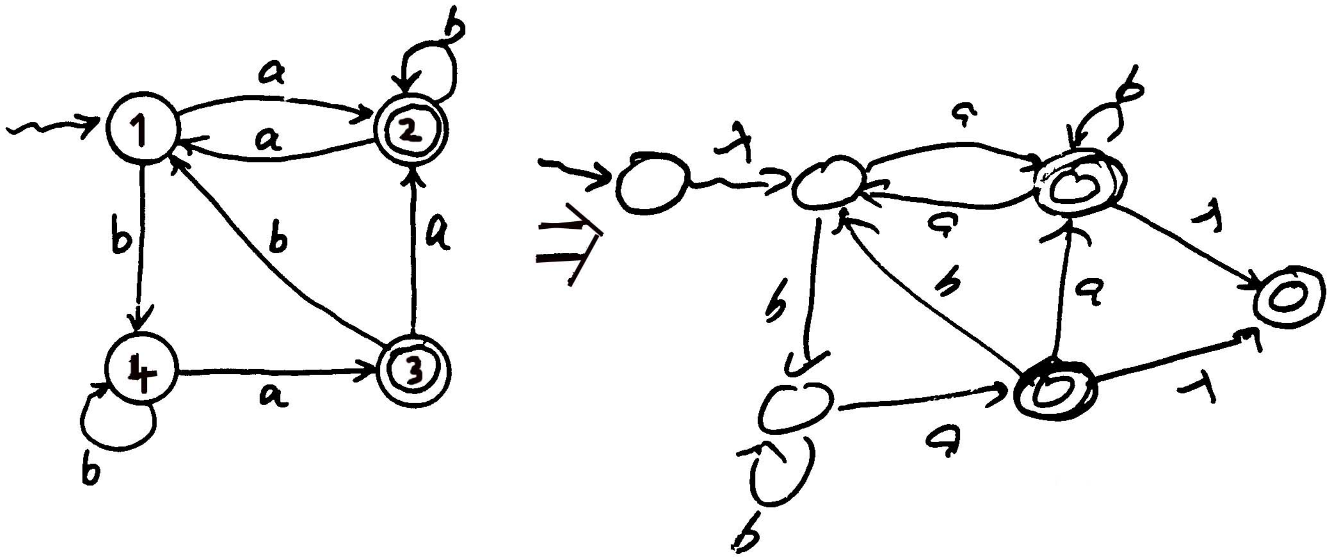
(ii) $\delta(p,q) = E$, **and**

(iii) $w \in L(E)$.

△ $\vdash^*$, $\vdash^+$ **are defined similarly as before.**

<u>Lemma</u> If M **is a DFA, Then there is an EFA** M' **with** $L(M') = L(M)$**.**

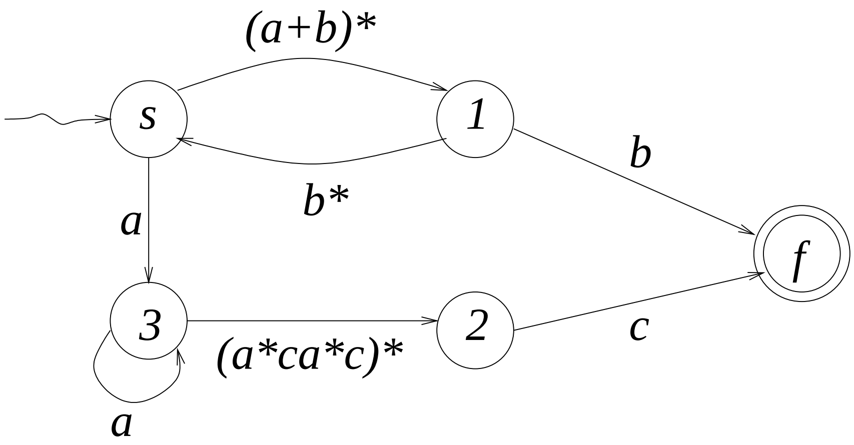
<u>Example</u> DFA into EFA.



Example:

An extended finite Automaton (EFA).

M:



Check if the following words are in $L(M)$

(1) $bbabab$

(2) $aabbc$

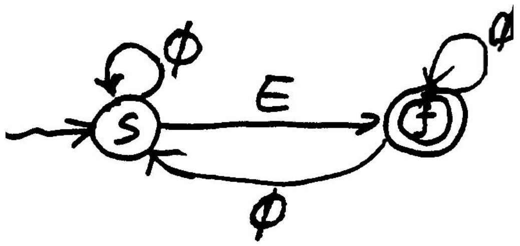
(3) $acccc$

(4) $aaaaac$

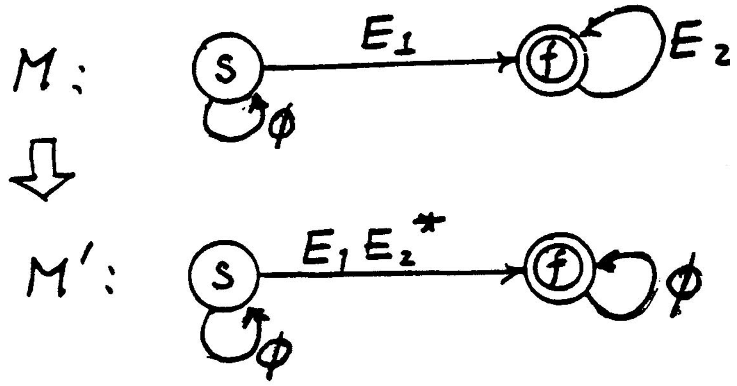
State Elimination Technique

Goal of the technique:

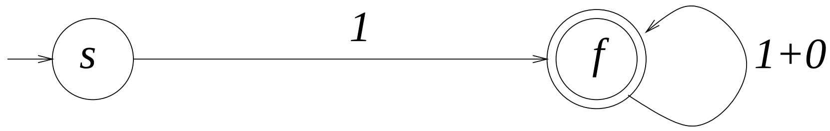
i.e.:



(1) EFA has 2 states



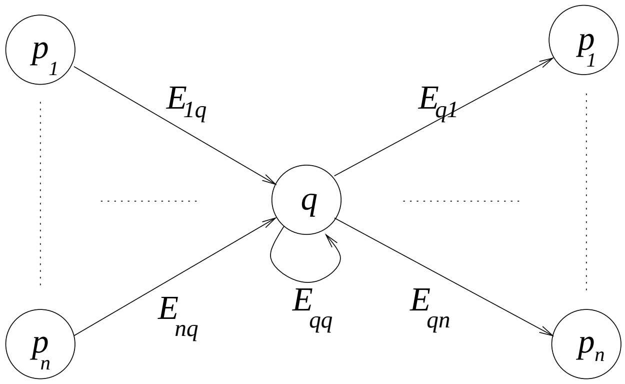
Example



(2) EFA M has $k+1$ states, $k \geq 2$.

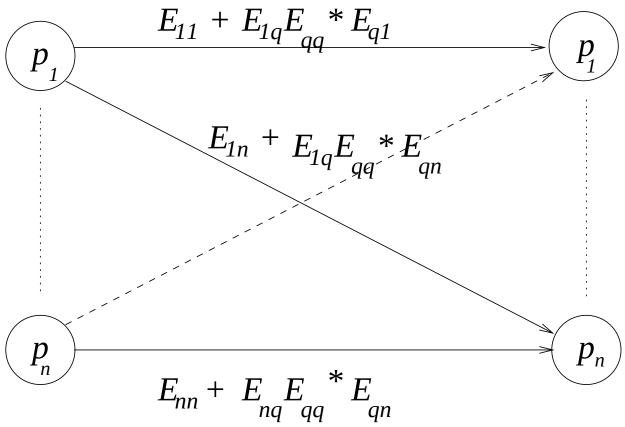
Then eliminate a state from M to form M':

M :



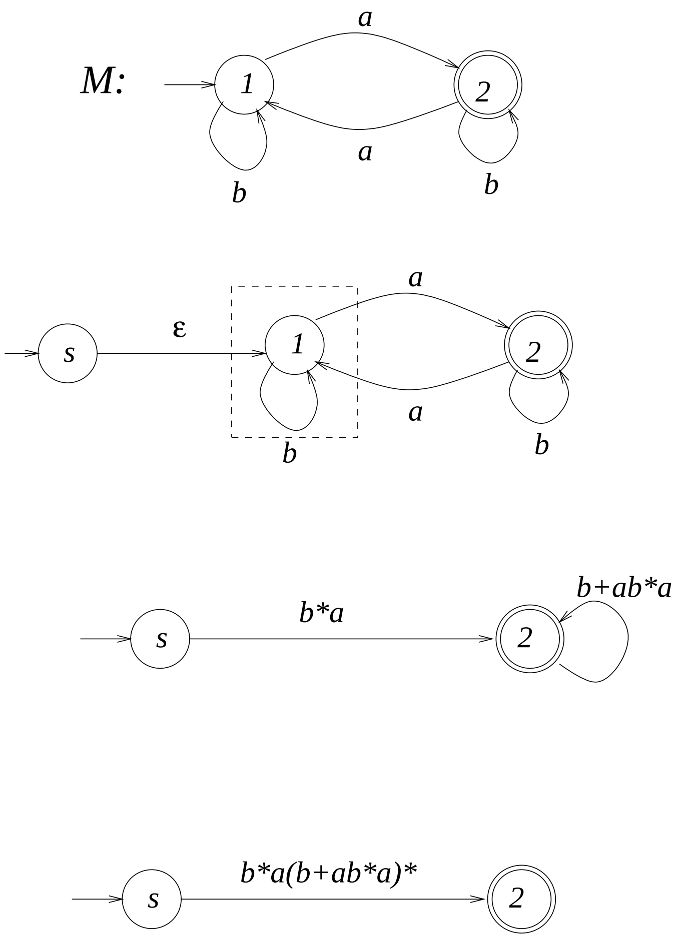
Note: $q \in Q - \{s,\ f\}$. **Consider all transitions** $(p_i,\ E_{iq},\ q)$ **and** $(q,\ E_{qi},\ p_i)$

M' :



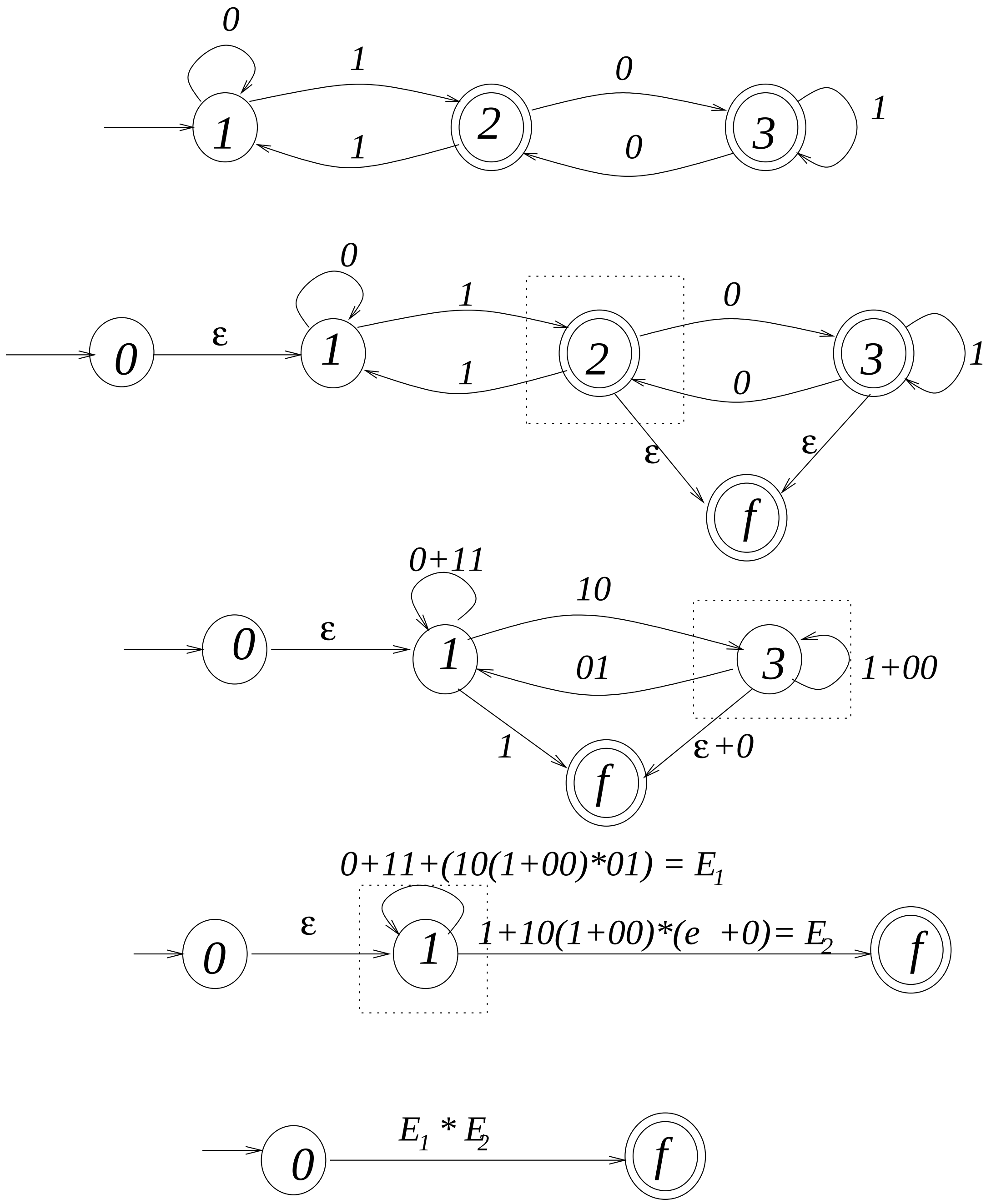
$$\delta'(p_i, p_j) = \delta(p_i, p_j) + \delta(p_i, q)(\delta(q, q))^* \delta(q, p_j)$$

Example



$$b^*a(b + ab^*a)^*$$

Example



Summary of the State Elimination Technique

(1) Add a new start state if the original one has incoming transitions.

(2) Add a new final state if there are more than one final states originally. Old final states become non-final states.

(3) Eliminate the states in $Q - \{s,\ f\}$ one by one.

(4) Eliminate the transition $\delta(f, f)$.