Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 36, S. 143–147 (1990)

ON A THEOREM OF GÜNTER ASSER

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1. Introduction

Recently, G. ASSER [2] has obtained two interesting characterizations of the class of unary primitive recursive string-functions over a fixed alphabet as Robinson algebras. Both characterizations use a somewhat artificial string-function, namely the string-function lexicographically associated with the number-theoretical excess-over-a-square function. Our aim is to offer two new and natural Robinson algebras which are equivalent to ASSER's algebras.

Let N denote the set of naturals, i.e. $\mathsf{N} = \{0, 1, 2, \ldots\}$, and $\mathsf{N}_+ = \mathsf{N} \setminus \{0\}$. We consider a fixed alphabet $\mathsf{A} = \{a_1, a_2, \ldots, a_r\}$, $r \geqq 2$, and denote by A^* the free monoid generated by A under concatenation (with e as the null-string). By length(w) we denote the length of the string w (length$(e) = 0$). For every $w \in \mathsf{A}^*$ and $m \in \mathsf{N}$ let $w^m = ww\ldots w$ (m times), in case $m > 0$, and $w^0 = e$. By Fnc (respectively, Fnc_A) we denote the set of all unary number-theoretical (respectively, string) functions. By I, $Succ$, E, C_m, Pd we denote the following number-theoretical functions: $I(x) = x$; $Succ(x) = x + 1$; $E(x) = x \dot{-} [\sqrt{x}]^2$; $C_m(x) = m$; $Pd(x) = x \dot{-} 1$, where $x \dot{-} y = \max(x - y, 0)$. By I^A, $Succ_i^\mathsf{A}$, C_u^A, σ, δ, π we denote the following string-functions: $I^\mathsf{A}(w) = w$; $Succ_i^\mathsf{A}(w) = wa_i$ $(1 \leqq i \leqq r)$; $C_u^\mathsf{A}(w) = u$; $\sigma(e) = a_1$, $\sigma(wa_i) = wa_{i+1}$, if $1 \leqq i < r$, and $\sigma(wa_r) = \sigma(w)\, a_1$; $\delta(e) = e$, $\delta(wa_i) = w$ $(1 \leqq i \leqq r)$; $\pi(e) = e$, $\pi(\sigma(w)) = w$. Furthermore we use the bijections $c\colon \mathsf{A}^* \to \mathsf{N}$, $\bar{c}\colon \mathsf{N} \to \mathsf{A}^*$ given by $c(e) = 0$, $c(wa_i) = r \cdot c(w) + i$, $1 \leqq i \leqq r$, and $\bar{c}(0) = e$, $\bar{c}(m + 1) = \sigma(\bar{c}(m))$; obviously $c(\bar{c}(m)) = m$ and $\bar{c}(c(w)) = w$.

To each $f \in Fnc$ we associate the string-function $s(f) \in Fnc_\mathsf{A}$ defined by $s(f)(w) = \bar{c}(f(c(w)))$; conversely, to each string-function g we associate the number-theoretical function $n(g) \in Fnc$ defined by $n(g)(x) = c(g(\bar{c}(x)))$. It is easily seen that for every $f \in Fnc$ and $g \in Fnc_\mathsf{A}$ one has $n(s(f)) = f$ and $s(n(g)) = g$. For example, $s(C_m) = C^\mathsf{A}_{\bar{c}(m)}$, $s(Succ) = \sigma$, $s(I) = I^\mathsf{A}$, $s(Pd) = \pi$.

For every $F \subseteqq Fnc$ and $G \subseteqq Fnc_\mathsf{A}$ we put $s(F) = \{s(f) \mid f \in F\}$ and $n(G) = \{n(g) \mid g \in G\}$. A mapping from Fnc^n in Fnc ($n \in \mathsf{N}_+$) is called an *n-ary operator in Fnc*, and analogous for Fnc_A. We consider the following operators in Fnc and Fnc_A:

$$sub(f, g) = h \quad \text{iff} \quad f, g, h \in Fnc \quad \text{and} \quad h(x) = f(g(x));$$

$$it_x(f) = h \quad \text{iff} \quad f, h \in Fnc \quad \text{and} \quad h(0) = x,\ h(y + 1) = f(h(y));$$

$$add(f, g) = h \quad \text{iff} \quad f, g, h \in Fnc \quad \text{and} \quad h(x) = f(x) + g(x);$$

$$diff(f, g) = h \quad \text{iff} \quad f, g, h \in Fnc \quad \text{and} \quad h(x) = f(x) \dot{-} g(x);$$

$$sub_\mathsf{A}(f, g) = h \quad \text{iff} \quad f, g, h \in Fnc_\mathsf{A} \quad \text{and} \quad h(w) = f(g(w));$$

$$\sigma\text{-}it_{\mathsf{A},u}(f) = h \quad \text{iff} \quad f, h \in Fnc_\mathsf{A} \quad \text{and} \quad h(e) = u,\ h(\sigma(w)) = f(h(w));$$

$$it_{\mathsf{A},u}(f_1, \ldots, f_r) = h \quad \text{iff} \quad f_1, \ldots, f_r,\ h \in Fnc_\mathsf{A} \quad \text{and} \quad h(e) = u,\ h(wa_i) = f_i(h(w)),\ 1 \leqq i \leqq r;$$

$$con_\mathsf{A}(f, g) = h \quad \text{iff} \quad f, g, h \in Fnc_\mathsf{A} \quad \text{and} \quad h(w) = f(w)\, g(w).$$

For every operator φ in *Fnc*,

$$s(\varphi)(f) = s(\varphi(n(f))), \text{ for every } f \in Fnc;$$

analogously, for every operator θ in *Fnc*,

$$n(\theta)(g) = n(\theta(s(g))), \text{ for every } g \in Fnc.$$

For example, $s(it_x) = \sigma\text{-}it_{A,c(x)}$, $n(\sigma - it_{A,w}) = it_{\bar{c}(w)}$.

Finally, for every subset $F \subseteq Fnc$ and every set X of operators in *Fnc*, $[F; X]$ denotes the smallest subset of *Fnc* which contains F and is closed under the operators belonging to X, and analogously for Fnc_A.

A simple, but useful, result in ASSER [2] establishes the following relations:

(1) *For every* $F \subseteq Fnc$ *and for every set* X *of operators in* Fnc, $s([F; X]) = [s(F); s(X)]$.

(2) *For every* $G \subseteq Fnc_A$ *and for every set* Y *of operators in* Fnc_A, $n([G; Y]) = [n(G); n(Y)]$.

2. Main results

The primitive recursive string-functions were introduced by ASSER [1] and studied by various authors (see EILENBERG and ELGOT [6], BRAINERD and LANDWEBER [4], CALUDE [5]). A famous result of R. M. ROBINSON [9] gives the following characterisation of the class $Prim^1$ of unary primitive recursive number-theoretical functions:

$$Prim^1 = [\{Succ, E\}; \{sub, it_0, add\}].$$

In ASSER [2] the following characterizations of the class $Prim_A^1$ of unary primitive recursive string-functions are obtained:

(3) $$Prim_A^1 = [\{\sigma, s(E)\}; \{sub_A, \sigma\text{-}it_{A,e}, s(add)\}],$$

(4) $$Prim_A^1 = [\{Succ_1^A, \ldots, Succ_r^A, s(E)\}; \{sub_A, it_{A,e}, con_A\}].$$

These characterizations use the somewhat artificial string-function $s(E)$ and the operator $s(add)$ in Fnc_A is also rather artificial.

In GEORGIEVA [7] (see also CALUDE [5]) one finds the following result:

$$Prim^1 = [\{Succ\}; \{sub, diff\} \cup \{it_x \mid x \in \mathrm{N}\}].$$

This formula can be simplified as follows:

(5) $$Prim^1 = [\{Succ\}; \{sub, diff, it_0\}].$$

All that remains to prove (5) is the inclusion

$$Prim^1 \subseteq P = [\{Succ\}; \{sub, diff, it_0\}],$$

i.e. the closure of P under the operators it_x for $x \in \mathrm{N}_+$. First we note that P contains the functions $sg = it_0(C_1)$, $\overline{sg} = diff(C_1, sg)$ and that P is closed under sum and product. Now let f in P and $h = it_x(f)$, $x \in \mathrm{N}_+$. If, for every natural $k > 0$, $f^k(x) = f(f(\ldots(f(x))\ldots)) \neq 0$ (k times), then $h = sub(h^*, Succ)$, where $h^* = it_0(g)$ and $g(y) = x \cdot \overline{sg}(y) + f(y) \cdot sg(y)$ ($\cdot$ denotes the product). In case there exists a natural $k > 0$ such that $f^k(x) = 0$, say the minimal one, then

$$h(y) = h^*(Succ(y)) \cdot \overline{sg}(t(x)) + h_*(t(x)) \cdot sg(t(x)),$$

where $t = diff(I, C_k)$, $h_* = it_0(f)$.

In view of (5) and (1), as a slight improved form of (3) we obtain:

(6) $Prim^1_A = [\{\sigma\}; \{sub_A, \sigma\text{-}it_{A,e}, s(diff)\}]$.

The string-function $s(E)$ is dropped, but the unpleasant operator $s(diff)$ in Fnc_A is still present. To overcome this difficulty we shall present our first result:

(7) $Prim^1_A = [\{Succ^A_1, \dots, Succ^A_r, \pi\}; \{sub_A, it_{A,e}, con_A\}]$.

For this, we denote by F the right-hand side of (7). In order to prove (7) we will show that (i) $\sigma \in F$, (ii) F is closed under $\sigma\text{-}it_{A,e}$, (iii) F is closed under $s(diff)$.

As in the proof of Proposition 2 in Asser [2] we begin with displaying a list of string-functions belonging to F:

a) $C^A_e(w) = e$: $C^A_e = it_{A,e}(\pi, \dots, \pi)$.

b) $\varkappa_i(w) = a_i$ $(1 \leqq i \leqq r)$: $\varkappa_i = sub_A(Succ^A_i, C^A_e)$.

c) $I^A(w) = w$: $I^A = it_{A,e}(Succ^A_1, \dots, Succ^A_r)$.

d) $sg^A_i(e) = e$, $sg^A_i(w) = a_i$, for $w \neq e$ $(1 \leqq i \leqq r)$: $sg^A_i = it_{A,e}(\varkappa_i, \dots, \varkappa_i)$.

e) $succ^A_i(w) = a_i w$ $(1 \leqq i \leqq r)$: $succ^A_i = con_A(\varkappa_i, I^A)$.

f) $mir(e) = e$, $mir(wu) = mir(u)\, mir(w)$: $mir = it_{A,e}(succ^A_1, \dots, succ^A_r)$.

g) $\lambda_i(w) = a_i^{\text{length}(w)}$ $(1 \leqq i \leqq r)$: $\lambda_i = it_{A,e}(Succ^A_i, \dots, Succ^A_i)$.

h) $\gamma_i(w) = a_i^{c(w)}$ $(1 \leqq i \leqq r)$:

$$\gamma_i = it_{A,e}(con_A(I^A \dots I^A, \varkappa_i), con_A(I^A \dots I^A, \varkappa_i\varkappa_i), \dots, con_A(I^A \dots I^A, \varkappa_i \dots \varkappa_i))$$

(the k-th place of the operator $it_{A,e}$ is $con_A(I^A \dots I^A, \varkappa_i \dots \varkappa_i)$ with r concatenations of I^A and k concatenations of $\varkappa_i$; $1 \leqq k \leqq r$).

i) $\alpha_i(w) = u$ iff $w = u a_i a_i \dots a_i$ and u does not terminate with a_i $(1 \leqq i \leqq r)$:

$$\alpha_i = sub_A(mir, sub_A(it_{A,e}(Succ^A_1, \dots, Succ^A_{i-1}, con_A(I^A, sg^A_i), Succ^A_{i+1}, \dots, Succ^A_r), mir)))\,.$$

j) $\beta_i(w) = a_i a_i \dots a_i$ iff $w = \alpha(w)\, a_i a_i \dots a_i$ $(1 \leqq i \leqq r)$: $\beta_i = it_{A,e}(C^A_e, \dots, Succ^A_i, \dots, C^A_e)$.

k) $\overline{sg}^A_i(e) = a_i$, $\overline{sg}^A_i(w) = e$, for $w \neq e$ $(1 \leqq i \leqq r)$:

$$\overline{sg}^A_i = sub_A(sg^A_i, sub_A(\beta_r, con_A(succ^A_r, sg^A_1)))\,.$$

l) $\psi_i(e) = a_{i+1}$, $\psi_i(w) = a_i$, for $w \neq e$ $(1 \leqq i < r)$: $\psi_i = con_A(\overline{sg}^A_{i+1}, sg^A_i)$;

$\psi_r(e) = a_1$, $\psi_r(w) = a_r$, for $w \neq e$: $\psi_r = con_A(\overline{sg}^A_1, sg^A_r)$.

m) $\chi(e) = e$, $\chi(ua_i) = ua_{i+1}$, for $1 \leqq i < r$, $\chi(ua_r) = ua_1$:

$$\chi = sub_A(mir, sub_A(it_{A,e}(con_A(I^A, \psi_1), \dots, con_A(I^A, \psi_r)), mir))\,.$$

We are now in a position to prove (i): $\sigma \in F$. Indeed,

$$\sigma = con_A(sub_A(\overline{sg}^A_1, \alpha_r), con_A(sub_A(\chi, \alpha_r), sub_A(\varkappa_1, \beta_r)))\,.$$

Passing to (ii) we note that

(8) $$\sigma\text{-}it_{A,e}(f) = sub_A(it_{A,e}(f,\dots,f),\gamma_1),$$

i.e. F is closed under the operator $\sigma\text{-}it_{A,e}$.

To finish the proof we recall that, for every $f \in Fnc_A$ and $n \in N_+$, $f^0 = I^A$ and $f^n = sub_A(f, sub_A(f,\dots, sub_A(f,f)\dots))$, n times. Using a double lexicographical induction one proves the equality

$$\pi^{c(w)}(u) = \bar{c}(diff(c(u),c(w))) \text{ for } u, w \in A^*,$$

which enables us to write the formula

(9) $$s(diff)(f,g) = sub_A(it_{A,e}(\sigma,\pi,I^A,\dots,I^A), con_A(sub_A(\gamma_1,f), sub_A(\gamma_2,g))),$$

for all f, $g \in Fnc_A$, thus proving (iii). This ends the proof of (7).

Our second Robinson algebra is the following:

(10) $$Prim^1_A = [\{\sigma,\pi\}; \{sub_A, it_{A,e}, con_A\}].$$

In view of (6) we must prove that the right-hand side of (10) is closed under $\sigma\text{-}it_{A,e}$ and $s(diff)$.

Again we proceed with displaying a sequence of primitive recursive string-functions belonging to the right-hand side of (10):

a) $I^A = sub_A(\pi,\sigma)$.

b) $C^A_e = it_{A,e}(\pi,\dots,\pi)$.

c) $\varkappa_i = sub_A(\sigma, sub_A(\sigma,\dots, sub_A(\sigma, C^A_e)))$,

where the operator sub_A appears i times $(1 \leqq i \leqq r)$.

d) $\gamma_i = it_{A,e}(con_A(M_r(I^A), M_1(\varkappa_i))\ con_A(M_r(I^A), M_2(\varkappa_i)),\dots, con_A(M_r(I^A), M_r(\varkappa_i)))$,

with $M_j(f) = con_A(f, con_A(f,\dots, con_A(f,f))\dots)$, where f is any string-function and the operator con_A appears $j \geqq 1$ times.

The proof of (10) is complete in view of (8) and (9).

Finally, we conjecture the validity of the following formula:

(11) $$Prim^1_A = [\{\sigma,\pi\}; \{sub_A, \sigma\text{-}it_{A,e}, con_A\}].$$

In view of (2) and $n(Succ^A_i)(x) = Succ^i(x)$, $1 \leqq i \leqq r$, (11) holds iff its right-hand side is closed under the operator $it_{A,e}$.

3. Final remarks

After finishing this paper we have learnt the following new characterizations of $Prim^1_A$ due to G. Asser [3]:

$$Prim^1_A = [\{Succ^A_1,\dots, Succ^A_r, \lambda\}; \{sub_A, it_{A,e}, con_A\}]$$
$$= [\{Succ^A_1,\dots, Succ^A_r, \varrho\}; \{sub_A, it_{A,e}, con_A\}],$$

where λ, ϱ are the component functions of the pairing function

$$\gamma(u, v) = a_1^{\mathrm{length}(u)} a_2 u v a_2 a_1^{\mathrm{length}(v)},$$

i.e., if $w = \gamma(u, v)$ for some strings u, v, then $\lambda(w) = u$ and $\varrho(w) = v$, else $\lambda(w) = \varrho(w) = e$.

Furthermore G. Asser (communication of July 13, 1989) has perceived that in (7) the function π can be replaced by the function δ, i.e.

$$(12) \qquad Prim_{\mathrm{A}}^{1} = \left[\left\{Succ_1^{\mathrm{A}}, \ldots, Succ_r^{\mathrm{A}}, \delta\right\}, \left\{sub_{\mathrm{A}}, it_{\mathrm{A},e}, con_{\mathrm{A}}\right\}\right].$$

The proof is essentially the same as for (7). Only a) must be replaced by

$$C_e^{\mathrm{A}} = it_{\mathrm{A},e}(\delta, \ldots, \delta),$$

and (9) must be replaced by

$$s(diff)(f, g)$$
$$= sub_{\mathrm{A}}\big(it_{\mathrm{A},e}(\sigma, \ldots, \sigma), sub_{\mathrm{A}}\big(it_{\mathrm{A},e}(Succ_1^{\mathrm{A}}, \delta, \ldots, \delta), (con_{\mathrm{A}}(sub_{\mathrm{A}}(\gamma_1, f), sub_{\mathrm{A}}(\gamma_2, g)))\big)\big).$$

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(Eingegangen am 14. November 1988)

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