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MORPHISMS PRESERVING DENSITIES*

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The notions of density, thinness, residue and ideal in a free monoid can all be expressed in terms of the infix order. Guided by these definitions we introduce the same notions with respect to arbitrary binary relations. We then investigate properties of these generalized notions and explore the connection to the theory of codes. We show that, under certain assumptions about the relation, density is preserved by an endomorphism or the inverse of an endomorphism if and only if – essentially – the endomorphism induces a permutation of the generators of the free monoid.

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C. R. Categories: F.4, G.2

1. INTRODUCTION

A language L – a set of words – over an alphabet X is said to be dense if every word u over X is the infix of some word v in L, that is, there are words x and y over X such that $v = xuy$. Suppose that X and Y are alphabets and that φ is a (homo-)morphism of the set X^* of words over X into the set Y^* of

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words over Y where the multiplication of words is their concatenation and where the empty word λ acts as an identity element. The morphism φ is said to preserve density if $\varphi(L)$ is dense for every dense language L over X; similarly, φ^{-1} preserves density if $\varphi^{-1}(L)$ is dense for every dense language over Y. It is a natural question to ask: which morphisms or inverse morphisms preserve density?

The case of $X=Y$ is of particular interest from the point of view of applications in language and coding theory as iteration of a morphism is a very common operation there – requiring $X=Y$ of course. Hence, in this paper, we nearly always assume that $X=Y$.

The case of $|X|<|Y|$ is ruled out by Proposition 6.6 below as there are no density preserving morphisms in this case. On the other hand, the case of $|X|>|Y|$ seems to be quite different from that of $|X|=|Y|$ – or $X=Y$ – and characterizing those morphisms that preserve density is an open problem.

A precursor to the present paper[1] answered this question. During the revisions[2] of that paper it was found that some of the main results had been proved independently in [12]. A careful analysis of the proofs indicated that an essentially identical characterization could be established for morphisms or inverse morphisms preserving other kinds of densities. These density notions arise naturally in the theory of codes (see [1]) as follows: Many interesting and useful classes of codes can be defined as classes of languages satisfying some independence property and, in many cases, this property can be expressed in terms of a binary relation. For example, the class of infix codes over the alphabet X consists of all languages L, such that $\lambda\notin L$ and no word $u\in L$ is an infix of a different word $v\in L$. Writing $u\leq_{\mathrm{i}} v$ to mean that u is an infix of v, the independence condition defining infix codes says:

$$\forall u,\ v\in L\ (u\leq_{\mathrm{i}} v\rightarrow u=v).$$

Similar characterizations by binary relations exist for many other classes of codes, some of which are provided further below.[3]

Given a binary relation ϱ on X^* the case of $\leq_{\mathrm{i}}=\varrho$ suggests mechanisms for the definition of density, residues, ideals, closure, independence and

[1]Written by L. K. and G. T. and accepted a few years ago by Semigroup Forum, but withdrawn (and hence not published) by the authors as some of the main results had been proved independently in [12].

[2]Now also involving H. J.

[3]A detailed discussion of this type of connection is provided in [2, 3]; for a summary see [1], where also error-detection and error-correction properties of such codes and their usability for information transmission over noisy channels are discussed.

maximality with respect to ϱ. These notions turn out to be meaningful beyond just their obvious rôles as generalizations. For example, in [11] and [4] *meta-constructions* are proposed to obtain maximal independent sets with respect to binary relations which expose the core properties of similar constructions known so far only for very few classes of codes.

After establishing the notation and reviewing some basic notions in Section 2 of this paper and briefly discussing the usual notion of density in Section 3, we define density *etc.*, with respect to an arbitrary binary relation ϱ and establish some of the basic properties of these notions in Section 4. As has already been noted in the context of the theory of codes (see [2, 3, 1]) the monoid structure of X^* plays no rôle in this part of the theory; hence the general theory is developed for relations on arbitrary sets. The interesting applications in the theories of languages and codes, of course, need to refer to the structure of X^*. This connection is indicated in Section 5. Section 6 contains the main results of this paper: Given ϱ, characterize the morphisms or inverse morphisms that preserve density with respect to ϱ. For many meaningful relations ϱ, these characterizations are simple – or even the same. Section 7 contains a few minor, but useful consequences. In Section 8 we present some conclusions.

A final remark in this Introduction: Some of the proofs in this paper may appear rather pedantic; we found that, at this level of generality, it was all too easy to jump to wrong conclusions just because they were *so very obviously true*. This made us build all the arguments in rather careful detail – admittedly at the risk of sometimes being "pedestrian". This attention to detail has eliminated several mistakes – that is, *things obviously or trivially true*, which were subtly wrong – and also helped removing unnecessary assumptions in many cases.

2. NOTATION AND BASIC NOTIONS

In this section we introduce the notation used throughout the paper and review some basic notions.

The symbol $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set S, let $|S|$ denote the cardinality of S and let 2^S denote the set of all subsets of S. Let S and T be sets and α a mapping of S into T. For a subset S' of S, $\alpha|_{S'}$ denotes the restriction of α to S'.

For a binary relation $\varrho \subseteq S \times T$, the set

$$\operatorname{dom} \varrho = \{s \mid s \in S,\ \exists t \in T\ (s, t) \in \varrho\}$$

is the *domain of* ϱ. Moreover,

$$\varrho^{-1} = \{(t,s) \mid (s,t) \in \varrho\}$$

is the *inverse of* ϱ, and, for $s \in S$,

$$\varrho(s) = \{t \mid t \in T,\ (s,t) \in \varrho\}.$$

Consequently,

$$\varrho^{-1}(t) = \{s \mid s \in S,\ (s,t) \in \varrho\}$$

for $t \in T$.

A *closure operator* on a set S is a mapping C of 2^S into 2^S with the following properties. For any L, $L' \subseteq S$ with $L \subseteq L'$ one has $CL \subseteq CL'$, $L \subseteq CL$, and $CCL = CL$.

Let X be an *alphabet*, that is, a finite non-empty set. Then X^* is the set of all *words* over X including the *empty word* λ. Let $w \in X^*$ and $a \in X$. Then $|w|_a$ is the number of occurrences of a in w and $|w| = \sum_{a \in X} |w|_a$ is the *length* of w. A *language* over X is a subset of X^*. For a language L, the *alphabet of* L, alph(L), is the set of all $a \in X$ with $|w|_a > 0$ for some $w \in L$.

A word $x \in X^*$ is said to be *primitive* if $x = y^n$ for $y \in X^*$ implies $n = 1$. Let Q be the set of all primitive words. For a word $x \in X^*$, let $\sqrt{x}$ denote the unique primitive word of which x is a power. For a language L, let $\sqrt{L} = \{\sqrt{x} \mid x \in L\}$.

Let x, $y \in X^*$. The *shuffle product* of x and y is the set

$$x \text{ III } y = \left\{ z \;\middle|\; \begin{array}{c} z \in X^*, \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n, y_1, \ldots, y_n \in X^* \\ x = x_1 \cdots x_n \wedge y = y_1 \cdots y_n \wedge z = x_1 y_1 \cdots x_n y_n \end{array} \right\},$$

that is, the shuffle product of x and y is the set of all words that can be obtained from x and y by shuffling them into each other while preserving the order of symbols in x and in y. For languages L_1, $L_2 \subseteq X^*$, the shuffle product is defined as

$$L_1 \text{ III } L_2 = \{x \text{ III } y \mid x \in L_1,\ y \in L_2\}.$$

Most of the results of this paper become trivial when $|X| = 1$ or, in this case, require extra, but trivial treatment. For this reason, we assume throughout this paper that all alphabets have at least 2 elements. Moreover, without loss of generality, we assume that the symbols a and b are distinct elements of any alphabet, unless this is explicitly or implicitly excluded.

3. DENSITIES

Let X be an alphabet. A language I is an *ideal* if $X^*IX^* \subseteq I$. An ideal I is *principal* if $I = X^*wX^*$ for some $w \in X^*$; in this case w is the *generator* of I.

A language L is said to be *dense* if, for every $u \in X^*$, there exist $x, y \in X^*$ such that $xuy \in L$. A language that is not dense is said to be *thin*. The *residue* of a language L is the set

$$W(L) = \{u \mid u \in X^*, \forall x, y \in X^*(xuy \notin L)\}.$$

Hence, L is dense if and only if $W(L) = \emptyset$. Equivalently, L is dense if its intersection with every principal ideal of X^* is non-empty.

When the alphabet X is a singleton set, then a language $L \subseteq X^*$ is dense if and only if L is infinite. To exclude such trivial cases, we assumed above that $|X| > 1$.

On X^* consider the *infix-order* $\leq_{\mathrm{i}}$ given by

$$x \leq_{\mathrm{i}} y \quad \text{if and only if } y \in X^*xX^*$$

for all $x, y \in X^*$. We re-express the notions of density, residue, and ideal using the infix-order. A language L is dense if and only if, for every $u \in X^*$, there is a $v \in L$ such that $u \leq_{\mathrm{i}} v$. The residue of L is the set

$$\{u \mid u \in X^*, \forall v \in L(u \not\leq_{\mathrm{i}} v)\}.$$

For a word $w \in X^*$, the ideal X^*wX^* generated by w is the set

$$\{v \mid v \in X^*,\ w \leq_{\mathrm{i}} v\}.$$

Thus, the definition of the notions of density, residue, and ideal follow a *general* schema, when defined in terms of a relation; this schema is to be explored in the rest of this paper. For some part of the analysis not even the multiplicative structure of X^* is relevant. Therefore, we introduce and discuss the basic structural properties simply on sets, turning back to free monoids only when their properties play a rôle. However, much of the analysis is motivated by questions arising in the theory of codes. Hence the reader might want to keep in mind concrete examples from this theory.

4. DENSITY, RESIDUE, IDEAL, CLOSURE, INDEPENDENCE, MAXIMALITY

For this section, let S be an arbitrary, but fixed, non-empty set. We introduce abstract notions of density, thinness, residue, ideal, closure, and maximality with respect to a binary relation on S. Then we derive some properties of these notions depending on basic properties of the relation, but independent of any structure of the set S.

DEFINITION 4.1 Let ϱ be a binary relation on S and let $L \subseteq S$.

(1) The set L is said to be *ϱ-dense* if, for every $x \in S$, there is a $y \in L$ such that $(x, y) \in \varrho$.
(2) The set L is *ϱ-thin* if it is not ϱ-dense.
(3) The *ϱ-residue* of L is the set

$$W_\varrho(L) = \{x \mid \forall y \in L \ (x, y) \notin \varrho\}.$$

(4) The set L is a *ϱ-ideal* if, for every $x \in L$ and every $y \in S$, the property $(x, y) \in \varrho$ implies $y \in L$.
(5) The *ϱ-closure* of L is the set

$$\mathbf{C}_\varrho L = \{y \mid y \in S, \ \exists x \in L \ (x, y) \in \varrho\}.$$

(6) The set L is a *principal ϱ-ideal* if it is an ideal and if there is an element $w \in L$ such that $L = \mathbf{C}_\varrho\{w\}$.
(7) The set L is *ϱ-independent* if, for any x, $y \in L$, $(x, y) \in \varrho$ implies $x = y$.
(8) The set L is *ϱ-maximal* if it is ϱ-independent and if no proper superset of L is ϱ-independent.

For $S = X^*$ and $\varrho = \leq_i$, the notions of ϱ-density, ϱ-thinness, ϱ-residue, and ϱ-ideal coincide with the usual ones of density, thinness, residue, and ideal; moreover, in this case the ϱ-closure of L is the ideal generated by L; hence, the principal ϱ-ideals are precisely the principal ideals; finally, for this choice of ϱ, the family of ϱ-independent sets is the family of infix codes (see [1] for details).

In the sequel, for a binary relation ϱ and x, $y \in S$ we use, interchangeably, the notations $(x, y) \in \varrho$ and $x \varrho y$. If the relation ϱ is the infix order $\leq_i$ on $S = X^*$, we use the terms *dense*, *thin*, *residue*, and *ideal* instead of $\leq_i$-dense, $\leq_i$-thin, $\leq_i$-residue and $\leq_i$-ideal; moreover, instead of $\leq_i$-closure we say *ideal generated by* ...

In the rest of this section we derive several elementary properties of the notions introduced in Definition 4.1.

LEMMA 4.2 *For any binary relation ϱ on S, the operator $\mathbf{C}_\varrho$ is monotonic.*

Proof Consider L, $L' \in S$ with $L \subseteq L'$ and $y \in \mathbf{C}_\varrho L$. Then there is $x \in L$ with $(x, y) \in \varrho$. By $L \subseteq L'$, $x \in L'$. Hence $y \in \mathbf{C}_\varrho L'$. ∎

LEMMA 4.3 *Let ϱ be a binary relation on S. The following statements are equivalent.*

(1) *The relation ϱ is transitive.*
(2) *For every set $L \subseteq S$, the set $\mathbf{C}_\varrho L$ is a ϱ-ideal.*
(3) *For every set $L \subseteq S$, the ϱ-residue of L is either empty or a ϱ-ideal.*

Proof First suppose ϱ is transitive. Consider $L \subseteq S$, $y \in \mathbf{C}_\varrho L$, and $z \in S$ such that $(y, z) \in \varrho$. There is an $x \in L$ such that $(x, y) \in \varrho$. By transitivity, $(x, z) \in \varrho$, hence $z \in \mathbf{C}_\varrho L$. This proves (2).

For (3), assume that $W_\varrho(L)$ is non-empty and consider $x \in W_\varrho(L)$ and $y \in S$ with $(x, y) \in \varrho$. Suppose $y \notin W_\varrho(L)$. Then there is a $z \in L$ with $(y, z) \in \varrho$. By transitivity, $(x, z) \in \varrho$, hence $x \notin W_\varrho(L)$, a contradiction. This proves (3).

For the converse, suppose that ϱ is not transitive. Then there are x, y, $z \in S$ such that $(x, y) \in \varrho$, $(y, z) \in \varrho$, and $(x, z) \notin \varrho$.

For (2), let $L = \{x\}$. Then $y \in \mathbf{C}_\varrho L$, but $z \notin \mathbf{C}_\varrho L$ and $\mathbf{C}_\varrho L$ is not a ϱ-ideal. For (3), let $L = \{z\}$. Then $x \in W_\varrho(L)$ and $y \notin W_\varrho(L)$, that is, $W_\varrho(L)$ is non-empty and not a ϱ-ideal. ■

LEMMA 4.4 *Let ϱ be a binary relation on S. The following statements hold true*:

(1) *If ϱ is reflexive then, for every set $L \subseteq S$, $L \subseteq \mathbf{C}_\varrho L$.*
(2) *If ϱ is transitive then, for every set $L \subseteq S$, $\mathbf{C}_\varrho \mathbf{C}_\varrho L \subseteq \mathbf{C}_\varrho L$ with equality when ϱ ϱ is reflexive.*
(3) *If ϱ is reflexive and transitive then $\mathbf{C}_\varrho$ is a closure operator. On the other hand, if ϱ is not transitive then $\mathbf{C}_\varrho$ is not a closure operator.*

Proof If ϱ is reflexive then $(x, x) \in \varrho$ for all $x \in S$, hence $L \subseteq \mathbf{C}_\varrho L$.

We now assume that ϱ is transitive. Consider $z \in \mathbf{C}_\varrho \mathbf{C}_\varrho L$. Then there is $y \in \mathbf{C}_\varrho L$ with $(y, z) \in \varrho$ and, consequently, there is $x \in L$ with $(x, y) \in \varrho$. By transitivity, $(x, z) \in \varrho$ and, therefore, $z \in \mathbf{C}_\varrho L$, that is, $\mathbf{C}_\varrho \mathbf{C}_\varrho L \subseteq \mathbf{C}_\varrho L$. If ϱ is also reflexive then $\mathbf{C}_\varrho L \subseteq \mathbf{C}_\varrho \mathbf{C}_\varrho L$.

Using Lemma 4.2, this proves that $\mathbf{C}_\varrho$ is a closure operator when ϱ is reflexive and transitive.

Finally, assume that ϱ is not transitive. Consider x, y, $z \in S$ with $(x, y) \in \varrho$, $(y, z) \in \varrho$, and $(x, z) \notin \varrho$. Let $L = \{x\}$. Then $z \notin \mathbf{C}_\varrho L$, but $z \in \mathbf{C}_\varrho \mathbf{C}_\varrho L$. ■

In Lemma 4.4, reflexivity is essentially only needed to establish that a set L is contained in its closure. One could, of course, avoid the assumption of reflexivity by changing Definition 4.1(5), the definition of the ϱ-closure, to include this condition. We opted for the simpler definition to make whichever assumptions would be needed explicit.

LEMMA 4.5 *Let ϱ be a binary relation on S. The following statements hold true.*

(1) *S is ϱ-dense if and only if dom $\varrho = S$.*
(2) *Let ϱ be reflexive. Then S is ϱ-dense and, for every set $L \subseteq S$, one has $W_\varrho(L) \cap L = \emptyset$.*

Proof For S to be ϱ-dense it is necessary and sufficient that, for every $x \in S$, there exists a $y \in S$ with $(x, y) \in \varrho$. This proves (1).

Assume ϱ is reflexive. Then, in particular, dom $\varrho = S$, hence S is ϱ-dense by (1). Now consider $x \in W_\varrho(L)$. Then there is no $y \in L$ such that $(x, y) \in \varrho$. As $(x, x) \in \varrho$, it follows that $x \notin L$. This proves (2). ∎

LEMMA 4.6

(1) *Let $L \subseteq S$ and let ϱ be a binary relation on S. The set L is ϱ-dense if and only if $W_\varrho(L) = \emptyset$.*
(2) *Let $L_1 \subseteq L_2 \subseteq S$ and let ϱ be a binary relation on S. Then $W_\varrho(L_2) \subseteq W_\varrho(L_1)$. If L_1 is ϱ-dense then L_2 is ϱ-dense.*
(3) *Let ϱ_1 and ϱ_2 be two binary relations on S such that $\varrho_1 \subseteq \varrho_2$ and let $L \subseteq S$. Then $W_{\varrho_2}(L) \subseteq W_{\varrho_1}(L)$. If L is ϱ_1-dense then it is ϱ_2-dense.*

Proof For the proof of (1), suppose that $W_\varrho(L)$ is non-empty. Consider $x \in W_\varrho(L)$. Then, for all $y \in L$, $(x, y) \notin \varrho$; hence, L is not ϱ-dense. Conversely, if L is not ϱ-dense then there is a element $x \in S$ such that $(x, y) \notin \varrho$ for all $y \in L$ and $W_\varrho(L)$ is non-empty.

For (2), consider $x \in W_\varrho(L_2)$. Then, for all $y \in L_2$, one has $(x, y) \notin \varrho$. As $L_1 \subseteq L_2$, one has $(x, y) \notin \varrho$ for all $y \in L_1$. Thus $x \in W_\varrho(L_1)$. If L_1 is ϱ-dense then $W_\varrho(L_1) = \emptyset$, hence $W_\varrho(L_2) = \emptyset$ and L_2 is ϱ-dense by (1).

For (3), consider $x \in W_{\varrho_2}(L)$. Then, for all $y \in L$, $(x, y) \notin \varrho_2$; hence $(x, y) \notin \varrho_1$, that is, $x \in W_{\varrho_1}(L)$. The remaining statement follows by (1). ∎

For any binary relation ϱ on a set S, let $\mathcal{D}_\varrho(S)$ be the family of all ϱ-dense sets in S. We write $\mathcal{D}_\varrho$ instead of $\mathcal{D}_\varrho(S)$ when S is understood.

LEMMA 4.7 *Let ϱ_1 and ϱ_2 be binary relations on S.*

(1) $\mathcal{D}_{\varrho_1 \cap \varrho_2} \subseteq \mathcal{D}_{\varrho_1} \cap \mathcal{D}_{\varrho_2}$.
(2) $\mathcal{D}_{\varrho_1} \cup \mathcal{D}_{\varrho_2} \subseteq \mathcal{D}_{\varrho_1 \cup \varrho_2}$.

Proof For (1), consider $L \in \mathcal{D}_{\varrho_1 \cap \varrho_2}$. Then, for every $u \in S$, there exists $v \in L$ such that $(u, v) \in \varrho_1 \cap \varrho_2$, that is, $(u, v) \in \varrho_1$ and $(u, v) \in \varrho_2$. This implies $L \in \mathcal{D}_{\varrho_1} \cap \mathcal{D}_{\varrho_2}$.

For the proof of (2), first consider $L \in \mathcal{D}_{\varrho_1} \cup \mathcal{D}_{\varrho_2}$. Let $u \in S$. There exists $v \in L$ such that $(u,v) \in \varrho_1$ or $(u,v) \in \varrho_2$, which implies $(u,v) \in \varrho_1 \cup \varrho_2$. Consequently, $L \in \mathcal{D}_{\varrho_1 \cup \varrho_2}$. ■

The inclusions stated in Lemma 4.7 are strict in general as proved by the following examples.

(1) Let $S = X^*$, $\varrho_1 = \{(w, wa) | w \in X^*\}$ and $\varrho_2 = \{(w, wa^2) | w \in X^*\}$. On the one hand, $\varrho_1 \cap \varrho_2 = \emptyset$ and, therefore, $\mathcal{D}_{\varrho_1 \cap \varrho_2} = \emptyset$. On the other hand,

$$\mathcal{D}_{\varrho_1} \cap \mathcal{D}_{\varrho_2} = \{L \mid L \subseteq X^*,\ X^*a \cup X^*a^2 \subseteq L\}$$

is non-empty.

(2) Let $S = \{a,b,c,d\}$ and $\varrho_1 = \{(a,b),(b,b)\}$, $\varrho_2 = \{(c,d),(d,d)\}$. Then $L = \{b,d\} \in \mathcal{D}_{\varrho_1 \cup \varrho_2}$ while $\mathcal{D}_{\varrho_1} \cup \mathcal{D}_{\varrho_2} = \emptyset$.

We now turn to some basic properties of ϱ-independence. Let $\mathcal{L}_\varrho$ be the set of ϱ-independent subsets of S. In the theory of languages and codes, independence is used to define classes of languages or codes; there the independence often models certain requirements for information transmission – for a few examples see the next section of this paper; details can be found, for instance, in [1].

LEMMA 4.8 *Let ϱ be a binary relation on S and let $L \subseteq S$. If $L \in \mathcal{L}_\varrho$ and L is ϱ-dense then L is ϱ-maximal.*

Proof Suppose L is not ϱ-maximal. Then there is an element $x \in S$, such that $x \notin L$ and $L \cup \{x\} \in \mathcal{L}_\varrho$. Hence, for all $y \in L$, $(x,y) \notin \varrho$ and $(y,x) \notin \varrho$. In particular, as $(x,y) \notin \varrho$ for all $y \in L$, L is not ϱ-dense. ■

The converse of Lemma 4.8 is not true in general. The next lemma establishes a sufficient condition for the converse conclusion and provides the hints for the construction of counter-examples for the general case.

LEMMA 4.9 *Let ϱ be a reflexive and symmetric binary relation on S. If $L \subseteq \mathcal{L}_\varrho$ is ϱ-maximal then L is ϱ-dense.*

Proof $L \subseteq \mathcal{L}_\varrho$ is ϱ-maximal – for any binary relation ϱ – if and only if, for all $x \in S$, there is a $y \in L$ such that $x = y$ or $(x,y) \in \varrho$ or $(y,x) \in \varrho$.

By reflexivity, if $x = y$ then $(x,y) \in \varrho$. By symmetry, if $(y,x) \in \varrho$ then $(x,y) \in \varrho$. Thus, L is ϱ-dense. ■

On the basis of Lemma 4.9 we now show by example that the converse of Lemma 4.8 is not true in general.

Example 4.10

(1) Let S be a set with at least two elements and let ϱ be a binary relation on S which is not reflexive. Consider $x \in S$ such that $(x,x) \notin \varrho$. Moreover, assume that $(x,y) \notin \varrho$ for all $y \in S$ and that there is a $z \in S \setminus \{x\}$ such that also $(z,x) \notin \varrho$. Such a set S and relation ϱ exist. For example, $S = \{x,z\}$ and $\varrho = \emptyset$ or $\varrho = \{(z,z)\}$ satisfy these assumptions. Let $L = \{x,z\}$. Then L is ϱ-independent, that is, $L \in \mathcal{L}_\varrho$. Let $L' \in \mathcal{L}_\varrho$ be ϱ-maximal with $L \subseteq L'$. The existence of L' is guaranteed by Zorn's lemma (see [1] for the details). As $x \neq z$ and $(x,y) \notin \varrho$ for all $y \in L'$, the set L' is not ϱ-dense.

(2) Let S be a 2-element set, say $S = \{x,y\}$. Let ϱ be a binary relation on S which is not symmetric; for example, let $(x,y) \notin \varrho$ and $(y,x) \in \varrho$. Such a relation ϱ exists. The set $L = \{y\}$ is ϱ-independent and even ϱ-maximal, but not ϱ-dense.

5. EXAMPLES: APPLICATION TO CODES AND LANGUAGES

The abstract notions of density, ideal, residue, independence introduced in Section 4 are suggested by constructs investigated in the theory of codes. In this case $S = X^*$ and ϱ is a binary relation on X^*. As candidates for ϱ we consider, in particular, the following relations, most of which play an important rôle in the definition of classes of codes or code-related languages [1, 9]. Some of these as well as the relations defined further below correspond to various error-detection capabilities of codes [1].

Example 5.1 Let w and v be arbitrary words in X^*.

(1) *Embedding order*: $w \leq_e v$ if and only if there exist $n \in \mathbb{N}_0$ and $w_1, \ldots, w_n$ and $v_0, v_1, \ldots, v_n$ in X^* such that $w = w_1 w_2 \ldots w_n$ and $v = v_0 w_1 v_1 w_2 \ldots w_n v_n$.
(2) *Length order*: $w \leq_u v$ if and only if $w = v$ or $|w| < |v|$.
(3) *Prefix order*: $w \leq_p v$ if and only if $v = wx$ for some $x \in X^*$.
(4) *Suffix order*: $w \leq_s v$ if and only if $v = xw$ for some $x \in X^*$.
(5) *Outfix relation*: $w \; \omega_o \; v$ if and only if there are w_1, u, $w_2 \in X^*$ such that $v = w_1 u w_2$ and $w = w_1 w_2$.
(6) *Infix order*: $w \leq_i v$ if and only if $v = xwy$ for some x, $y \in X^*$.
(7) *Division order*: $w \leq_d v$ if and only if $v = wx = yw$ for some x, $y \in X^*$.
(8) *Commutation order*: $w \leq_c v$ if and only if $v = xw = wx$ for some $x \in X^*$.
(9) *Power order*: $w \leq_f v$ if and only if $v = w^n$ for some $n \geq 1$.

All the relations of Example 5.1 except ω_o are partial orders. The relation ω_o is not transitive. Its transitive closure is the embedding order. There are many more special relations of interest in the context of coding theory [1, 9]. As before, for a binary relation ϱ one considers the class $\mathcal{L}_\varrho$ of *ϱ-independent* languages. For ϱ according to Example 5.1(1)–(8), the classes $\mathcal{L}_\varrho$ are the classes of hypercodes, block codes (or uniform codes),[4] prefix codes, suffix codes, outfix codes, infix codes, 2-ps-codes, and 2-codes. Some of the languages in the classes of 2-ps-codes and 2-codes are not codes in the usual sense (see [1] for details). The class of $\leq_{\mathrm{f}}$-independent languages is a proper superset of the class of 2-codes as the language $\{ababab, abab\}$ is $\leq_{\mathrm{f}}$-independent, but not a 2-code, while, on the other hand, every 2-code is $\leq_{\mathrm{f}}$-independent.

If a language L in $\mathcal{L}_\varrho$ is ϱ-dense then this means in essence that – with the use of L for information transmission over noisy channels in mind – L makes very good use of the set X^* of all possible words; by Lemma 4.8, no words can be added to L without violating the condition of ϱ-independence.

While the assumption of reflexivity is not problematic in the context of the theory of codes (with a few exceptions), assuming symmetry is clearly unacceptable in that context as most of the important classes of codes would be excluded. Thus, Example 4.10 and Lemma 4.9 explain the basic reasons why, in the context of the theory of codes, maximality cannot usually be expected to imply density.

The relations of Example 5.1 are ordered by inclusion as follows:

$$\leq_{\mathrm{f}} \subsetneq \leq_{\mathrm{c}} \subsetneq \leq_{\mathrm{d}} \subsetneq \left\{ \begin{matrix} \leq_{\mathrm{p}} \\ \leq_{\mathrm{s}} \end{matrix} \right\} \subsetneq \left\{ \begin{matrix} \leq_{\mathrm{i}} \\ \omega_o \end{matrix} \right\} \subsetneq \leq_{\mathrm{e}} \subsetneq \leq_{\mathrm{u}}.$$

We also consider the infinite chain

$$\omega_{\mathrm{i}_1} \subsetneq \omega_{\mathrm{i}_2} \subsetneq \cdots \subsetneq \omega_{\mathrm{i}_n} \subsetneq \cdots \subsetneq \leq_{\mathrm{e}}$$

of binary relations such that $\omega_{\mathrm{i}_1} = \leq_i$ and $\leq_{\mathrm{i}} \cup \omega_o \subsetneq \omega_{\mathrm{i}_n}$ for $n > 1$ which is defined as follows.

DEFINITION 5.2 Let $n \in \mathbb{N}$. For u, $v \in X^*$ let $(u, v) \in \omega_{\mathrm{i}_n}$ if and only if

$$\exists u_1, u_2, \ldots, u_n, v_0, v_1, \ldots, v_n \; (u = u_1 u_2 \cdots u_n \wedge v = v_0 u_1 v_1 u_2 \cdots u_n v_n).$$

These relations are a natural generalization of the prefix, suffix, infix and outfix relations. They were introduced in [10] and independently, together with three further related chains,[5] in [5, 6, 8, 7]. For a summary see [1],

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[4]Hence the subscript 'u' in $\leq_{\mathrm{u}}$.

[5]These three chains, while interesting in the context of codes, do not add to the present considerations as they are interleaved with the chain of the relations ω_{i_n}.

p. 552–553. There the class $\mathcal{L}_{\mathrm{i}_n}=\mathcal{L}_{\omega_{\mathrm{i}_n}}$ is called the class of *infix-shuffle codes of index n* or of *i_n-codes*. Note that

$$\lim_{n\to\infty} \omega_{\mathrm{i}_n} = \bigcup_{n=1}^{\infty} \omega_{\mathrm{i}_n} = \le_{\mathrm{e}}.$$

The relations ω_{i_n} are reflexive and anti-symmetric, but not transitive for $n>1$. Their transitive closure is the relation $\le_{\mathrm{e}}$. The following example shows that, as a consequence of their non-transitivity, the ω_{i_n}-residue of an ω_{i_n}-thin language is not an ω_{i_n}-ideal in general.

Example 5.3 Let $X=\{a,b\}$ and $L=\{ababa\}$. Then $a^3\in W_{\omega_{\mathrm{i}_2}}$, $(a^3,aba^2)\in \omega_{\mathrm{i}_2}$, $(aba^2,ababa)\in\omega_{\mathrm{i}_2}$, hence $aba^2\notin W_{\omega_{\mathrm{i}_2}}$, that is, $W_{\omega_{\mathrm{i}_2}}$ is not an ω_{i_2}-ideal. Similar examples can be constructed for every $n>2$. The construction is based on the proof idea of Lemma 4.3.

Finally, let $\omega_{\mathrm{b}}=\le_{\mathrm{p}}\cup\le_{\mathrm{s}}$. The class $\mathcal{L}_{\omega_{\mathrm{b}}}$ is the class of bifix codes (see [1,9]). The relation ω_{b} is reflexive and anti-symmetric, but not transitive; its transitive closure is the infix order.

PROPOSITION 5.4 *A language L is ω_o-dense if and only if it is ω_b-dense.*

Proof Assume that L is ω_{o}-dense. Consider $u\in X^*$ and let $v=uu$. There exists $x\in X^*$ such that $v_1xv_2\in L$ were $v=v_1v_2$. If $|v_1|\le|u|$, then $|v_2|\ge|u|$ and $v_2=yu$. If $x'=v_1xy$, then $x'u\in L$. If $|v_1|\ge|u|$, then similarly there exists x'' such that $ux''\in L$. Therefore L is ω_{b}-dense.

Conversely, assume that L is ω_{b}-dense. We have $\omega_{\mathrm{b}}\subseteq\omega_{\mathrm{o}}$; therefore, L is ω_{o}-dense by Lemma 4.6(3). ■

To add some concrete intuition, we briefly discuss the notions of ideal and residue for some of the relations of Example 5.1. Let $L\subseteq X^*$.

The language L is a $\le_{\mathrm{p}}$-ideal if and only if it is a right ideal. Dually, it is a $\le_{\mathrm{s}}$-ideal if and only if it is a left ideal. The $\le_{\mathrm{p}}$-residue of L is the complement of the set of prefixes of words in L.

Let n be the minimal length of a word in L. The language L is a $\le_{\mathrm{u}}$-ideal if and only if $L=L\cup X^{n+1}X^*$. The $\le_{\mathrm{u}}$-residue of L is non-empty if and only if L is finite and, in this case, it consists of all the words that are strictly longer than the longest word in L.

The language L is a $\le_{\mathrm{e}}$-ideal if and only if $L=L \text{ III } X^*$. The $\le_{\mathrm{e}}$-residue of L is the set W with $(W \text{ III } X^*)\cap L=\emptyset$.

The language L is a $\le_{\mathrm{r}}$-ideal if and only if $L=L^{(+)}$. If L is a $\le_{\mathrm{r}}$-ideal then $W_{\le_{\mathrm{r}}}(L)=(Q\backslash\sqrt{L})^{(+)}$. Indeed, consider $x\in W_{\le_{\mathrm{r}}}(L)$. Then, for all $y\in L$ and all $n\in\mathbb{N}$, $y\ne x^n$. We first show that no power of $\sqrt{x}$ is in L.

Assume, on the contrary, that there is a $y \in L$ and an $m \in \mathbb{N}$ such that $y = (\sqrt{x})^m$ and let $x = (\sqrt{x})^k$. Then $(\sqrt{x})^{km} = x^m = y^k$. As L is a $\leq_r$-ideal, $y^k \in L$ and y^k is a power of x, a contradiction; this proves that, for all $y \in L$ and all $m \in \mathbb{N}$, $y \neq (\sqrt{x})^m$. Therefore, for all $y \in L$, $\sqrt{y} \neq \sqrt{x}$, that is, $\sqrt{x} \in Q \backslash \sqrt{L}$ and $x \in (Q \backslash \sqrt{L})^{(+)}$. Conversely, if $x \in (Q \backslash \sqrt{L})^{(+)}$ then $\sqrt{x} \notin \sqrt{L}$. Hence, for all $y \in L$ and $n \in \mathbb{N}$, $y \neq (\sqrt{x})^n$ and this implies $x^+ \cap L = \emptyset$.

We conclude this section with a result that relates ϱ-ideals and ϱ-density for reflexive relations ϱ satisfying the following condition:

For any non-empty language $L \subseteq X^*$ which is not ϱ-dense and for any $x \in L$ there is a $y \in X^* \backslash L$ such that $(x, y) \in \varrho$.

We call a relation ϱ with this property an extensive relation.

PROPOSITION 5.5 *Let ϱ be a reflexive and extensive binary relation on X^*. Then the following statements hold true.*

(1) *Every non-empty ϱ-ideal is ϱ-dense.*
(2) *The complement of any ϱ-thin language is ϱ-dense.*

Proof Consider a non-empty ϱ-ideal L. Suppose L is not ϱ-dense. Then $L \neq X^*$ by Lemma 4.5; hence, the set $M = X^* \backslash L$ is neither empty nor the whole set X^*. Let $x \in L$. As ϱ is extensive, there is $y \in M$ such that $(x, y) \in \varrho$. As L is a ϱ-ideal, $y \in L$. But $L \cap M = \emptyset$, a contradiction. This proves (1).

Now, for the proof of (2), consider a ϱ-thin language $L \subseteq X^*$ and let $M = X^* \backslash L$. If $L = \emptyset$ then $M = X^*$, and M is ϱ-dense by Lemma 4.5 as ϱ is reflexive. Thus, we may assume that $L \neq \emptyset$. If $M = \emptyset$ then $L = X^*$, and L is ϱ-dense by Lemma 4.5, a contradiction. Therefore, also $M \neq \emptyset$. By Lemma 4.5, $W_\varrho(L) \subseteq M$.

Suppose M is ϱ-thin. Consider $x \in X^*$ such that, for all $y \in X^*$, $(x, y) \in \varrho$ implies $y \notin L$. As L is ϱ-thin such an x exists. Then $x \in W_\varrho(L)$, hence $x \in M$. As ϱ is extensive, there is an element $y \in X^* \backslash M$ such that $(x, y) \in \varrho$. But $X^* \backslash M = L$, hence $y \in L$, a contradiction. Thus, M is not ϱ-thin. ■

PROPOSITION 5.6

(1) *If ϱ is a binary relation containing ω_b then ϱ is extensive.*
(2) *If ϱ_1 and ϱ_2 are binary relations on X^* such that $\varrho_1 \subseteq \varrho_2$ and if ϱ_1 is extensive then also ϱ_2 is extensive.*

Proof To prove (1), let L be a non-empty ϱ-thin language. Thus $M = X^* \backslash L$ is non-empty and not equal to X^*. Consider $x \in L$ and $y \in W_\varrho(L)$. Then $z \in L$ implies $(y, z) \notin \varrho$ and, in particular, $y \not\leq_p z$ and $y \not\leq_s z$ as $\omega_b = \leq_p \cup \leq_s \subseteq \varrho$. Therefore, $xy \notin L$. On the other hand $x \leq_p xy$, hence $(x, xy) \in \varrho$. This shows that ϱ is extensive.

To prove (2), suppose ϱ_1 is extensive and ϱ_2 is not extensive. Then there is a language L which is ϱ_2-thin such that, for some $x \in L$ and all $y \in X^* \setminus L$, $(x,y) \notin \varrho_2$. Then, by Lemma 4.6(3), L is also ϱ_1-thin. Moreover, $\varrho_1 \subseteq \varrho_2$ implies that $(x,y) \notin \varrho_1$ for all $y \in X^* \setminus L$. Thus ϱ_1 is not extensive, a contradiction. ■

COROLLARY 5.7 *Let ϱ be a reflexive binary relation on X^* such that $\omega_b \subseteq \varrho$. The following statements hold true.*

(1) *Every non-empty ϱ-ideal is ϱ-dense.*
(2) *If $L \subseteq X^*$ is ϱ-thin then $X^* \setminus L$ is ϱ-dense.*

Proof By Proposition 5.6 the relation ϱ is extensive. Hence the statements hold true by Proposition 5.5. ■

Corollary 5.7 implies, in particular, that ϱ-ideals for

$$\varrho \in \{\omega_b, \omega_o, \leq_i, \leq_e, \leq_u\} \cup \{\omega_{i_n} \mid n > 1\}$$

are ϱ-dense as these relations are extensive. On the other hand, the relation $\leq_c$ is not extensive. To see this, let p be a primitive word and let $L = X^+ \setminus p^+$. Consider $x \in L$ and $y \in X^* \setminus L = p^*$. Then $x \leq_c y$ implies $\sqrt{x} = \sqrt{y}$, which is impossible. By Proposition 5.6(2), also $\leq_f$ is not extensive.

The relation $\leq_p$ is not extensive. For example, consider $X = \{a, b\}$ and $L = aX^*$. Then L is a $\leq_p$-ideal. As $b \not\leq_p v$ for all $v \in L$, the language L is not $\leq_p$-dense. As $\leq_p$ is reflexive it follows from Proposition 5.5(1) that the relation is not extensive. A similar argument shows that not every $\leq_d$-ideal is $\leq_d$-dense, hence $\leq_d$ is not extensive.

6. MORPHISMS AND INVERSE MORPHISMS PRESERVING DENSITIES

An endomorphism α of X^* is said to *preserve ϱ-density* if, for any $L \subseteq X^*$, $\alpha(L)$ is ϱ-dense whenever L is ϱ-dense; similarly, α^{-1} is said to *preserve ϱ-density* if, for any $L \subseteq X^*$, $\alpha^{-1}(L)$ is ϱ-dense whenever L is ϱ-dense. The core problem of this paper is as follows:

Under which conditions does an endomorphism or the inverse of an endomorphism of X^* preserve ϱ-density? This study started with the following result, concerning density in the usual sense, found independently also in [12].

PROPOSITION 6.1 *Let L be a dense language over the alphabet X, and let α be an endomorphism of X^*. Then $\alpha(L)$ is dense if and only if $\alpha|_X$ is a permutation of X.*

Although the proof of Proposition 6.1 will be superseded by the proof of a more general result, we present it here because it exhibits some essential structural features.

Proof[6] If $\alpha|_X$ is a permutation then $\alpha(L)$ is dense.

For the converse implication, let $m = 2\max\{|\alpha(a)| \mid a \in X\}$. If $m = 0$ then $\alpha(L) = \lambda$, which is not dense. Therefore, $m > 0$.

As $\alpha(L)$ is dense, for each $b \in X$, there exist $u, v \in X^*$ such that $ub^m v \in \alpha(L)$. The alphabet X being finite, there exists a unique $a \in X$ such that $\alpha(a) = b^k$ for some positive integer k.

Suppose that $\alpha(a) = b^k$ and $k \neq 1$ for some a, $b \in X$. Take $c \neq b$, $c \in X$. Hence $X^*cbcX^* \cap \alpha(L) = \emptyset$ and $\alpha(L)$ is not dense, a contradiction. Therefore $\alpha(a) = b$ for all $b \in X$ and $\alpha(X) = X$. ■

Keeping in mind that density can be defined using the infix order, careful examination of the proof shows that very little of it depends on the infix order *per se*. Indeed, the idea of this proof can be carried over to the following far more general situation. To state the result, we need one auxiliary definition.

DEFINITION 6.2 Let ϱ be a binary relation on X^* and let α be an endomorphism of X^*.

(1) The relation ϱ is *compatible* with α if, for all x, $y \in X^*$, the inclusion $(x, y) \in \varrho$ implies $(\alpha(x), \alpha(y)) \in \varrho$.
(2) The relation ϱ is *compatible* with α^{-1} if, for all $x, y \in X^*$ and $x' \in \alpha^{-1}(x)$, $y' \in \alpha^{-1}(y)$, $(x, y) \in \varrho$ implies $(x', y') \in \varrho$.

All the relations listed in Example 5.1, except $\leq_u$, are compatible with any endomorphism of X^*. The relation $\leq_u$ is compatible with an endomorphism[7] α of X^* if and only if $\alpha(X) \subseteq X^n$ for some $n \in \mathbb{N}_0$. Compatibility with α^{-1} is quite different. For each of the relations ϱ of Example 5.1 there is an endomorphism α of X^* such that ϱ is not compatible with α^{-1}.

[6]This proof is essentially just a paraphrase of the one of [12].

[7]If $\alpha(X) \subseteq X^n$ then $|\alpha(x)| = n|x|$ for every $x \in X^*$; this implies that α is compatible with $\leq_u$. For the converse, assume that there are x, $y \in X$ such that $|\alpha(x)| < |\alpha(y)|$. Let $n = |\alpha(x)|$, $m = |\alpha(y)|$. If $n = 0$ then $y \leq_u x^2$, but $\alpha(y) \not\leq_u \alpha(x^2) = \lambda$. Hence, assume that $n > 0$ and let $k = 2n + 1$, and $l = 2m$. Then $k < l$ and, therefore, $y^k \leq_u x^l$. On the other hand, $|\alpha(y^k)| = km = 2mn + m > ln = |\alpha(x^l)|$, hence $\alpha(y^k) \not\leq_u \alpha(x^l)$. This proves that α is not compatible with α.

Consider $X=\{a,b\}$ and $\alpha(a)=ab$, $\alpha(b)=abab$. Let $u=ab$ and $v=abab$. Then $u\le_f v$, hence $(u,v)\in\varrho$ for any relation ϱ of Example 5.1. On the other hand, $\alpha^{-1}(u)=\{a\}$ and $\alpha^{-1}(v)=\{a^2,b\}$ and $(a,b)\notin\varrho$ for any of these relations. The relation $\le_p$ is compatible with α^{-1} if and only if $|\alpha(X)|=|X|$ and $\alpha(X)$ is a prefix code.[8] Similarly, the relation $\le_u$ is compatible with α^{-1} if and only if $|\alpha(X)|=|X|$ and $\alpha(X)$ is a block (or uniform) code. For the other relations of Example 5.1 similar, but more complicated conditions arise. The following observation is needed further below.

LEMMA 6.3 *Let α be an automorphism of X^*. Then each of the relations in Example 5.1, each ω_{i_n} and also ω_b is compatible with α^{-1}.*

Proof As α is an automorphism, also α^{-1} is an automorphism. Each of the relations is compatible with automorphisms. ∎

LEMMA 6.4 *Let α be an endomorphism of X^* and let ϱ_1 and ϱ_2 be binary relations on X^*. If $\varrho_1\subseteq\varrho_2$ and α preserves ϱ_1-density then α preserves ϱ_2-density.*

Proof Consider a ϱ_1-dense language L. By Lemma 4.6 L is also ϱ_2-dense. As $\alpha(L)$ is ϱ_1-dense, $\alpha(L)$ is also ϱ_2-dense by Lemma 4.6. ∎

THEOREM 6.5 *Let α be an endomorphism of X^* and let ϱ be a binary relation on X^*. The following statements hold true.*

(1) *If ϱ is reflexive and $\alpha|_X$ is a permutation of X then $\alpha(X^*)$ is ϱ-dense.*
(2) *If ϱ is transitive and compatible with α and if $\alpha(X^*)$ is ϱ-dense then, for every $L\subseteq X^*$ which is ϱ-dense, also $\alpha(L)$ is ϱ-dense.*
(3) *If there is an $L\subseteq X^*$ such that $\alpha(L)$ is ϱ-dense, then $\alpha(X^*)$ is ϱ-dense.*
(4) *If $\varrho\subseteq\omega_{i_n}$ for some $n\in\mathbb{N}$ and $\alpha(X^*)$ is ϱ-dense then $\alpha|_X$ is a permutation of X.*

Proof For (1), as $\alpha|_X$ is a permutation of X, α is an automorphism of X^*, hence $\alpha(X^*)=X^*$. Thus, for any $x\in X^*$, one has $x\in\alpha(X^*)$ and $(x,x)\in\varrho$ by reflexivity.

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[8]First suppose that $\alpha(X)$ is a prefix code with $|\alpha(X)|=|X|$ and consider $x,y\in\alpha(X^*)$ such that $x\le_p y$. It follows that α is injective; hence, instead of considering α^{-1} as a relation, we can consider it as a mapping of $\alpha(X^*)$ onto X^*. If $x=\lambda$ then $\alpha^{-1}(x)=\lambda\le_p\alpha^{-1}(y)$. Hence assume that $x\ne\lambda$. As $\alpha(X)$ is a prefix code there are unique words $x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m}\in\alpha(X)$ such that $x=x_1\cdots x_n$ and $y=xx_{n+1}\cdots x_{n+m}$. Hence $\alpha^{-1}(x)\le_p\alpha^{-1}(y)$. Conversely, if $\alpha(X)$ is not a prefix code then there are distinct x', $y'\in X$ such that $\alpha(x')\le_p\alpha(y')$, but $x'\not\le_p y'$. Similarly, if $|\alpha(X)|\ne|X|$, then there distinct x', $y'\in X$ such that $\alpha(x')=\alpha(y')$, hence again $\alpha(x')\le_p\alpha(y')$, but $x'\not\le_p y'$.

For the proof of (2), consider $x \in X^*$. As $\alpha(X^*)$ is ϱ-dense, there is $z \in \alpha(X^*)$ with $(x,z) \in \varrho$. Let $z' \in \alpha^{-1}(z)$. As L is ϱ-dense, there is $y' \in L$ with $(z',y') \in \varrho$. Let $y = \alpha(y')$. Hence $y \in \alpha(L)$ and, by compatibility, $(z,y) \in \varrho$. Transitivity implies $(x,y) \in \varrho$.

(3) Let $L \subseteq X^*$ be such that $\alpha(L)$ is ϱ-dense. Since $\alpha(L) \subseteq \alpha(X^*)$, then Lemma 4.6(2) implies that $\alpha(X^*)$ is also ϱ-dense.

(4) Let $m = \max\{|\alpha(x)| \ |x \in X\}$ and $t = 2n(m+1)$. For $a \in X$, consider a^t. As $\alpha(X^*)$ is ϱ-dense, there is $v \in \alpha(X^*)$ such that $(a^t, v) \in \varrho \subseteq \omega_{\mathrm{i}_n}$. Thus, there are $t_1, \ldots, t_n$ and $v_0, v_1, v_2, \ldots, v_n \in X^*$ such that $t_1 + \cdots + t_n = t$ and $v_0 a^{t_1} v_1 \cdots a^{t_n} v_n = v$. There is an i such that $t_i \geq 2(m+1)$ and therefore, as $v \in \alpha(X^*)$ there is $b_a \in X$ such that $\alpha(b_a) = a^{k(a)}$ for some k, $1 \leq k(a) \leq m$. As this is true for every a, b_a is uniquely determined by a.

Thus, for every $a \in X$ there is a positive integer $k(a)$ and a $b_a \in X$ such that $\alpha(b_a) = a^{k(a)}$. Both b_a and $k(a)$ are uniquely determined, that is, the correspondences $a \mapsto b_a$ and $a \mapsto k(a)$ are mappings. Moreover, as X is finite and α is a mapping, the mapping $a \mapsto b_a$ is a permutation of X; hence, for every $b \in X$, $\alpha(b)$ is a power of some $a \in X$ and $b = b_a$. We now show that the inverse of the mapping $a \mapsto b_a$ is equal to $\alpha|_X$, that is, that $k(a) = 1$ for all $a \in X$.

Assume that $k(a) > 1$ for some $a \in X$. As $|X| \geq 2$, there is $c \in X \setminus \{a\}$. Consider the word $u = (ca)^n c$. As $\alpha(X^*)$ is ϱ-dense, there is $v \in \alpha(X^*)$ such that $(u,v) \in \varrho \subseteq \omega_{\mathrm{i}_n}$. Hence $u = u_1 \cdots u_n$ for some $u_i \in X^*$ and $v = v_0 u_1 v_1 \cdots u_n v_n$ for some $v_i \in X^*$. Thus, there is an i such that $cac \leq_{\mathrm{i}} u_i \leq_{\mathrm{i}} v$. This is impossible as $\alpha(b)$ is a power of an element of X for every $b \in X$ and $k(a) > 1$. Hence, it follows that $k(a) = 1$ for all $a \in X$.

This completes the proof for $m > 0$. Assume $m = 0$. Then $\alpha(X^*) = \{\lambda\}$ which is never ϱ-dense. Hence $m = 0$ is impossible. ■

One of the main arguments in the proof of Theorem 6.5(4) can be extended to morphisms involving two different alphabets as follows.

PROPOSITION 6.6 *Let X and Y be alphabets, let $\alpha: X^* \to Y^*$ be a morphism. Let ϱ be a binary relation on Y^* contained in ω_{i_n} – considered on Y^* – for some $n \in \mathbb{N}$. If $\alpha(X^*)$ is ϱ-dense then the following statements hold true:*

(1) *For every $a \in Y$ there is an element $b \in X$ and a positive integer $k_{a,b}$ such that $\alpha(b) = a^{k_{a,b}}$.*
(2) $|Y| \leq |X|$.

Proof As in the proof of Theorem 6.5(4), define m to be the maximal length of a word in $\alpha(X)$ and let $t = 2n(m+1)$. Again the case of $m = 0$ is impossible.

For $a \in Y$, consider a^t. As $\alpha(X^*)$ is ϱ-dense there is a $v \in \alpha(X^*)$ with $(a^t, v) \in \varrho$. By $\varrho \subseteq \omega_{i_n}$, for some $v_0, v_1, \ldots, v_n \in Y^*$ and some $t_1, \ldots, t_n$ one has $v = v_0 a^{t_1} v_1 \cdots a^{t_n} v_n$ and $t = t_1 + \cdots + t_n$. Hence, for at least one t_i, $t_i \geq 2(m+1)$. The fact that $v \in \alpha(X^*)$ implies that there is a $b \in X$ and a $k_{a,b} \in \mathbb{N}$ with $\alpha(b) = a^{k_{a,b}}$. This proves (1).

If $|X| < |Y|$ then (1) is impossible. This proves (2). ■

Thus, with ϱ as in Proposition 6.6, $\alpha(X^*)$ is never ϱ-dense when $|X| < |Y|$. The case of $|X| = |Y|$ is covered by in essence Theorem 6.5. For the case of $|X| > |Y|$ we conjectured that there is an alphabet $\bar{X} \subseteq X$ such that $\alpha|_{\bar{X}}$ is a bijection of $\bar{X}$ onto Y. This is not true as shown by the following example.

Example 6.7 Let $X = \{a, b, c\}$ and $Y = \{x, y\}$. Consider the morphism $\alpha: X^* \to Y^*$ given by $\alpha(a) = x^2$, $\alpha(b) = y$ and $\alpha(c) = xy$. Let $\varrho = \leq_i$. Every word $u \in Y^*$ has the form

$$u = x^{n_1} y^{m_1} \cdots x^{n_k} y^{m_k}$$

with n_1, $m_k \geq 0$ and $n_2, \ldots, n_k$, $m_1, \ldots, m_{k-1} > 0$. For an integer n, let $\pi(n) = 0$ if n is even and $\pi(n) = 1$ if n is odd. Let $v_u \in X^*$ be the word

$$v_u = a^{\lfloor n_1/2 \rfloor} c^{\pi(n_1)} b^{m_1 - \pi(n_1)} a^{\lfloor n_2/2 \rfloor} c^{\pi(n_2)} b^{m_2 - \pi(n_2)} \cdots a^{\lfloor n_k/2 \rfloor} c^{\pi(n_k)} b^{m_k}.$$

Then $u \leq_i \alpha(v_u)$. Thus $\alpha(X^*)$ is $\leq_i$-dense.

We now derive several immediate consequences of Theorem 6.5.

COROLLARY 6.8 *Let α be an endomorphism of X^* and let ϱ be a reflexive and transitive binary relation, compatible with α, such that $\varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$. The following statements are equivalent.*

(1) *$\alpha|_X$ is a permutation of X.*
(2) *$\alpha(X^*)$ is ϱ-dense.*
(3) *$\alpha(L)$ is ϱ-dense for some ϱ-dense language L.*
(4) *$\alpha(L)$ is ϱ-dense for all ϱ-dense languages L.*
(5) *$\alpha(X^*)$ is dense.*
(6) *$\alpha(L)$ is dense for some dense language L.*
(7) *$\alpha(L)$ is dense for all dense languages L.*

Proof Statement (1) implies (2) by Theorem 6.5(1). Statements (3) and (4) follow from (2) by Theorem 6.5(2). Statement (3) and, hence, also Statement (4) implies (2) by Theorem 6.5(3). The equivalence of (1) and (6) is stated in Proposition 6.1. Statement (1) implies (5) and (7) by Theorem 6.5(1,2). Statement (1) is implied by (5) or (7) by Proposition 6.1; it is implied by (2) because of Theorem 6.5(4). ■

COROLLARY 6.9 *Let α be an endomorphism of X^*. The following statements are equivalent.*

(1) *$\alpha|_X$ is a permutation of X.*
(2) *α preserves ϱ-density for any $\varrho \in \{\leq_f, \leq_c, \leq_d, \leq_p, \leq_s, \leq_i\}$.*

COROLLARY 6.10 *Let α be an endomorphism of X^* and let ϱ be a reflexive relation such that $\varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$. The following statements are equivalent.*

(1) *$\alpha|_X$ is a permutation of X.*
(2) *$\alpha(X^*)$ is ϱ-dense.*
(3) *$\alpha(X^*)$ is dense.*

From Corollary 6.10 we cannot conclude anything about ϱ-density preservation of languages other than X^*, as, in general, transitivity of ϱ would be required. The transitive closure of any reflexive relation ϱ satisfying $\omega_o \subseteq \varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$, however, is the embedding order $\leq_e$; and for the embedding order Corollary 6.10 does not hold as shown further below.

COROLLARY 6.11 *Let ϱ be a reflexive and extensive binary relation on X^* with $\varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$. For an endomorphism α of X^*, $\alpha(X^*)$ is a ϱ-ideal if and only if $\alpha|_X$ is a permutation of X.*

Proof Being a ϱ-ideal, $\alpha(X^*)$ is ϱ-dense by Proposition 5.5. By Theorem 6.5(4), $\alpha|_X$ is a permutation of X. This implies that $\alpha(X^*) = X^*$. Clearly, X^* is a ϱ-ideal. ∎

A language $L \subseteq X^*$ is said to be *shuffle-dense* if it is $\leq_e$-dense. The *shuffle-residue* of L is the set

$$W_{sh}(L) = W_{\leq_e} = \{u \mid u \in X^*,\ (u \text{ III } X^*) \cap L = \emptyset\}.$$

The language L is *shuffle-thin* if and only if $W_{sh}(L) \neq \emptyset$.

While, according to Theorem 6.5(4), for every $n \in \mathbb{N}$, the ω_{i_n}-density of $\alpha(X^*)$ is equivalent to $\alpha|_X$ being a permutation of X, this result is not preserved as $n \to \infty$. For example, let $X = \{a, b\}$ with $\alpha(a) = a^2$, $\alpha(b) = b^2$. The language $\alpha(X^*) = \{a^2, b^2\}^*$ is not ω_{i_n}-dense[9] for any $n \in \mathbb{N}$, but it is shuffle-dense. The condition for shuffle-density to be preserved is much weaker.

LEMMA 6.12 *A submonoid $S \subseteq X^*$ is shuffle-dense if and only if alph$(S) = X$.*

[9]To see this take $u = (ab)^{n+1}$. Any factorization of u into n factors contains at least one word u_i of length greater than 2, that is, a factor u_i containing aba or bab. Thus, if $(u, v) \in \omega_{i_n}$ then $aba \leq_i v$ or $bab \leq_i v$, which is impossible for $v \in \alpha(X^*)$.

Proof Assume that the submonoid $S \subseteq X^*$ is shuffle-dense. If $a \in X$, then $(a \text{ III } x) \cap S \neq \emptyset$ for some $x \in X^*$ and, hence, $a \in \text{alph}(S)$.

Conversely, assume that the submonoid S satisfies $\text{alph}(S) = X$. Let $u \in X^*$ and suppose that $u = a_1 a_2 \cdots a_k$ with $a_i \in X$. Since $a_i \in alph(S)$, then $x_i a_i y_i \in S$ for some $x_i, y_i \in X^*$. Hence $w = x_1 a_1 y_1 \cdots x_k a_k y_k \in S$. Let $v = x_1 y_1 \cdots x_k y_k$. Then $w \in u \text{ III } v$, thus $u \leq_e w$ and $w \in S$. Thus, S is shuffle-dense. ■

THEOREM 6.13 *Let α be an endomorphism of X^*. The following statements are equivalent.*

(1) *$alph(\alpha(X^*)) = X$.*
(2) *$\alpha(X^*)$ is shuffle-dense.*
(3) *If $L \subseteq X^*$ is shuffle-dense then also $\alpha(L)$ is shuffle-dense.*

Proof As X^* is shuffle-dense, (3) implies (2). Moreover, as $\alpha(X^*)$ is a submonoid of X^*, statements (1) and (2) are equivalent by Lemma 6.12.

Now consider a shuffle-dense language L, hence $L \neq \emptyset$, and assume that $\text{alph}(\alpha(X^*)) = X$. Clearly, $\lambda \leq_e w$ for any $w \in L$. Consider $u = a_1 a_2 \cdots a_k \in X^+$ with $a_i \in X$. As $\text{alph}(\alpha(X^*)) = \text{alph}(\alpha(X)) = X$, there are $x_i, y_i \in X^*$ and $b_i \in X$ for $i = 1, \ldots, k$ such that $\alpha(b_i) = x_i a_i y_i$. Let $v = b_1 b_2 \cdots b_k$. As L is shuffle-dense, there is $z \in L$ such that $v \leq_e z$. The word z has the form

$$z = z_0 b_1 z_1 b_2 z_2 \cdots b_k z_k$$

with $z_i \in X^*$ for $i = 0, \ldots, k$. Hence

$$\begin{aligned}\alpha(z) &= \alpha(z_0)\alpha(b_1)\alpha(z_1)\alpha(b_2)\alpha(z_2)\cdots\alpha(b_k)\alpha(z_k)\\ &= \alpha(z_0)x_1a_1y_1\alpha(z_1)x_2a_2y_2\alpha(z_2)\cdots x_ka_ky_k\alpha(z_k) \in \alpha(L),\end{aligned}$$

that is, $u \leq_e \alpha(z)$. Therefore, $\alpha(L)$ is shuffle-dense. ■

For a language $L \subseteq X^*$, let

$$\text{alph}_\infty(L) = \{a \mid a \in X, \ \forall n \in \mathbb{N} \ \exists w \in L \ |w|_a > n\}.$$

The set $\text{alph}_\infty(L)$ consists of those elements of X which occur in unbounded numbers in words in L; for its complement, the set $X \setminus \text{alph}_\infty(L)$, there is an integer n such that $|w|_a \leq n$ for every $w \in L$ and every $a \in X \setminus \text{alph}_\infty(L)$. The preservation of $\leq_u$-density is characterized in terms of $\text{alph}_\infty(L)$. Recall that a language is $\leq_u$-dense if and only if it is infinite. The following lemma is probably well-known.

LEMMA 6.14 *Let α be an endomorphism of X^* and $L \subseteq X^*$ be an infinite language. The following statements are equivalent.*

(1) $\alpha(alph_\infty(L)) \neq \{\lambda\}$.
(2) $\alpha(L)$ *is infinite.*

Proof If $\alpha(\text{alph}_\infty(L)) \neq \{\lambda\}$ then $\text{alph}_\infty(\alpha(L)) \neq \emptyset$, hence $\alpha(L)$ is infinite. On the other hand, assume that $\alpha(\text{alph}_\infty(L)) = \{\lambda\}$.

Consider the endomorphism α_1 of X^* defined by

$$\alpha_1(a) = \begin{cases} \lambda, & \text{if } a \in \text{alph}_\infty(L), \\ a, & \text{if } a \in X \setminus \text{alph}_\infty(L), \end{cases}$$

for all $a \in X$ and the endomorphism α_2 of X^* defined by

$$\alpha_2(a) = \begin{cases} a, & \text{if } a \in \text{alph}_\infty(L), \\ \alpha(a), & \text{if } a \in X \setminus \text{alph}_\infty(L), \end{cases}$$

for all $a \in X$. Then, for every $w \in X^*$, $\alpha(w) = \alpha_2(\alpha_1(w))$. Moreover, alph $(\alpha_1(L)) = X \setminus \text{alph}_\infty(L)$. This proves that $\alpha_1(L)$ is finite and, therefore, also $\alpha(L)$ is finite. ∎

PROPOSITION 6.15 *Let α be an endomorphism of X^*. The following statements are equivalent.*

(1) α *preserves* $\leq_u$*-density.*
(2) $\alpha(L)$ *is infinite for every infinite language* $L \subseteq X^*$.
(3) $\alpha(a^*)$ *is infinite for every* $a \in X$.
(4) $\alpha(a) \neq \lambda$ *for every* $a \in X$.

Proof In view of Lemma 6.14 it suffices to observe that $\alpha(\text{alph}_\infty(L)) \neq \{\lambda\}$ for every infinite $L \subseteq X^*$ if and only if $\alpha(a^*)$ is infinite for every $a \in X$. ∎

We now turn to the question of which kind of endomorphisms α have the property that their inverses α^{-1} preserve densities. We start with a set of examples showing that certain natural conjectures fail to be true.

Example 6.16 Let $X = \{a, b\}$.

(1) Let α be the endomorphism of X^* defined by $\alpha(a) = \alpha(b) = \lambda$. Then $\alpha^{-1}(L) = X^*$ for any language L with $\lambda \in L$. Thus, $\alpha^{-1}(L)$ can be a dense language when L is not. However, α^{-1} does not preserve density as $\alpha^{-1}(X^+) = \emptyset$ is thin whereas X^+ is dense.
(2) Consider the language

$$L = \{w \mid w \in X^*,\ |w|_a = |w|_b\}.$$

The language L is dense. Let the endomorphism α be defined by $\alpha(a) = \alpha(b) = a^2$. Then $\alpha^{-1}(b^*) = \emptyset$ and $\alpha^{-1}(L) = \{\lambda\}$. Thus, the pre-images of both the thin language b^* and the dense language L are thin.
(3) Let L be the language of (2), and let $\alpha(a) = a^2$ and $\alpha(b) = b^2$. Then $\alpha^{-1}(L) = L$ is dense.
(4) Consider the language L of (2) and the injective endomorphism α defined by $\alpha(a) = a^2$ and $\alpha(b) = ab$. Then, $\alpha^{-1}(L) = b^*$, that is, the pre-image of a dense language is thin.

The phenomena exposed in Example 6.16 suggest that, for α^{-1} to preserve density, we should again consider the condition of $\alpha|_X$ being a permutation of X. Moreover, with some limitations this turns out to carry over to the more general issue of preserving ϱ-density.

THEOREM 6.17 *Let α be an endomorphism of X^*. Then α^{-1} preserves density if and only if $\alpha|_X$ is a permutation of X.*

Proof If $\alpha|_X$ is a permutation of X then α^{-1} is an endomorphism of X^* and preserves density by Proposition 6.1.

Now consider an endomorphism α of X^* such that α^{-1} preserves density. Note first that $\alpha(X) \neq \{\lambda\}$. Indeed, otherwise $\alpha^{-1}(X^+) = \emptyset$, contradicting the assumption that α^{-1} preserves density.

Assume that $\alpha|_X$ is not a permutation of X. In this case, $\alpha(X^*) \subsetneq X^*$ is not dense by Proposition 6.1. Let $T = X^* \setminus \alpha(X^*)$. According to Proposition 5.5(2), T is nonempty and dense. Now

$$\alpha^{-1}(T) = \alpha^{-1}(T \cap \alpha(X^*)) = \alpha^{-1}(\emptyset) = \emptyset$$

which is not dense, a contradiction. ∎

For a generalization of Theorem 6.17 to relations different from the infix order, the results of Proposition 5.5 turn out to be crucial.

THEOREM 6.18 *Let ϱ be a binary relation on X^* and let α be an endomorphism of X^*. The following statements hold true.*

(1) *If $\varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$, ϱ is extensive and reflexive, X^+ is ϱ-dense, and if α^{-1} preserves ϱ-density then $\alpha|_X$ is permutation of X.*
(2) *If ϱ is compatible with α^{-1} and if $\alpha|_X$ is a permutation of X then α^{-1} preserves ϱ-density.*

Proof For the proof of (1), assume that α^{-1} preserves ϱ-density. If $\alpha(X) = \{\lambda\}$ then $\alpha^{-1}(X^+) = \emptyset$. As X^+ is ϱ-dense and $\emptyset$ is not, this case is excluded. Therefore, $\alpha(X) \neq \{\lambda\}$.

Now assume that $\alpha|_X$ is not a permutation of X. By Theorem 6.5(4), $\alpha(X^*)$ is not ϱ-dense. By Lemma 4.5, as ϱ is reflexive, X^* is ϱ-dense, hence $\alpha(X^*)\subsetneq X^*$. By Proposition 5.5(2), as ϱ is reflexive and extensive, $X^*\setminus \alpha(X^*)$ is ϱ-dense. But $\alpha^{-1}(X^*\setminus \alpha(X^*))$ is empty, hence not ϱ-dense, a contradiction.

We turn to the proof of (2). Suppose $\alpha|_X$ is a permutation of X. Then α is an automorphism of X^*; consequently, for every $y\in X^*$, there is a unique $z\in X^*$ such that $\alpha^{-1}(y)=\{z\}$. Let L be ϱ-dense. Consider $x\in X^*$. Then there is $y\in L$ such that $(\alpha(x),y)\in\varrho$. Let z be the unique element of $\alpha^{-1}(y)$; hence $z\in\alpha^{-1}(L)$. Moreover $\alpha^{-1}(\alpha(x))=\{x\}$. As ϱ is compatible with α^{-1}, $(x,z)\in\varrho$. Thus $\alpha^{-1}(L)$ is ϱ-dense. ■

Thus, surprisingly, for α and α^{-1} the situation is quite similar. The endomorphism α or its inverse preserve ϱ-density if and only if $\alpha|_X$ is a permutation of X – provided some conditions are satisfied, and the sets conditions are nearly the same.

- For the case of α, it suffices that ϱ be reflexive, transitive, contained in ω_{i_n} for some n, and compatible with α.
- For the case of α^{-1}, it suffices that ϱ be reflexive, extensive, contained in ω_{i_n} for some n, and compatible with α^{-1}.

The discussion following Definition 6.2 indicates that the condition of ϱ being compatible with α^{-1} may be the hardest to satisfy.

In the proof of Theorem 6.18 we use the fact that $\alpha^{-1}(S)$ is empty for some ϱ-dense set $S\subseteq X^*$. In some sense, this is a rather trivial situation. Hence, the proof of Theorem 6.18 suggests that it could be useful to exclude such *trivial* cases. The following modified notion could be interesting to explore: Let α be an endomorphism of X^*. We say that α^{-1} *weakly preserves ϱ-density* if $\alpha^{-1}(L)$ is either empty or ϱ-dense for every ϱ-dense language $L\subseteq X^*$.

7. MORPHISMS, THIN LANGUAGES AND IDEALS

The preceding section answered the question as to which endomorphisms preserve the density of a language. The property of languages to be thin behaves quite differently under morphisms. Let ϱ be a binary relation on X^* and let α be an endomorphism of X^*. We say that α *preserves ϱ-thinness* if $\alpha(L)$ is ϱ-thin whenever $L\subseteq X^*$ is ϱ-thin; Similarly, α^{-1} is said to *preserve ϱ-thinness* if $\alpha^{-1}(L)$ is ϱ-thin whenever L is ϱ-thin. As is to be expected, many more endomorphisms preserve ϱ-thinness than ϱ-density.

THEOREM 7.1 *Let $\varrho \subseteq \omega_{i_n}$ for some $n \in \mathbb{N}$, and let α be an endomorphism of X^*. The following statements hold true.*

(1) *If ϱ is compatible with α^{-1} then α preserves ϱ-thinness.*
(2) *If ϱ is one of the relations ω_o, ω_b, $\leq_i$, $\leq_d$, $\leq_c$, $\leq_f$ or ω_{i_n} for some $n \in \mathbb{N}$ then α preserves ϱ-thinness.*
(3) *When $\alpha|_X$ is not a permutation of X then $\alpha(L)$ is ϱ-thin for every $L \subseteq X^*$.*

Proof To prove (1) and (2), assume that $L \subseteq X^*$ is ϱ-thin and $\alpha(L)$ is ϱ-dense. By Lemma 4.6(2), $\alpha(X^*)$ is ϱ-dense. By Theorem 6.5(4), $\alpha|_X$ is a permutation of X. This implies $L = \alpha^{-1}(\alpha(L))$. Consider $x \in X^*$. Then there is $y \in \alpha(L)$ such that $(\alpha(x), y) \in \varrho$. Let z be the unique word such that $\alpha^{-1}(y) = \{z\}$; hence, $z \in L$. Moreover, $\alpha^{-1}(\alpha(x)) = \{x\}$.

For (1), as ϱ is compatible with α^{-1}, $(x, z) \in \varrho$. Hence, L is ϱ-dense, a contradiction. For (2), the statement follows by Lemma 6.3.

For the proof of (3), assume that $\alpha|_X$ is not a permutation of X. Consider $L \subseteq X^*$ and assume that $\alpha(L)$ is ϱ-dense. By Lemma 4.6(2), also $\alpha(X^*)$ is ϱ-dense. By Theorem 6.5(4), $\alpha|_X$ is a permutation of X, a contradiction. ■

The following example shows that a mere inclusion of ϱ in one of the relations listed in Theorem 7.1(2) is not sufficient in general; without equality, one might not be able to conclude $(x, y) \in \varrho$ from $(\alpha(x), \alpha(y)) \in \varrho$ even when $\alpha|_X$ is a permutation of X.

Example 7.2 Let $X = \{a, b\}$ and let $\alpha(a) = b$, $\alpha(b) = a$. Let

$$\varrho = \{(b^k, b^{k+r}) \mid k \in \mathbb{N},\ r \in \mathbb{N}_0\} \cup \{(a^k, a^k) \mid k \in \mathbb{N}_0\}.$$

Thus $\varrho \subseteq\ \leq_i = \omega_{i_1}$. Then $(\alpha(a^k), \alpha(a^{k+r})) \in \varrho$, but $(a^k, a^{k+r}) \notin \varrho$ for $k, r \in \mathbb{N}$. A set L is ϱ-thin if and only if $a^* \not\subseteq L$ or $b^+ \cap L$ is finite. When L is ϱ-thin, $b^+ \not\subseteq \alpha(L)$ or $a^+ \cap \alpha(L)$ is finite. Hence, for instance,

$$L = \{a^{2n} \mid n \in \mathbb{N}\} \cup b^*$$

is ϱ-thin while $\alpha(L)$ is ϱ-dense.

8. CONCLUDING REMARKS

Using the schema suggested by the definition of ideals we have defined the notions of density and thinness – and a few related ones – for arbitrary binary relations. For endomorphisms and relations bounded by the shuffle relations we have obtained a complete characterization of density-preservation and a partial one of thinness-preservation. When the source alphabet is smaller than the target alphabet, density is not preserved. When it is larger, very little is known.

Most of our results apply to relations bounded by the shuffle relations, but do not hold when taking the limit of the shuffle relations, that is using the embedding order. We established some preliminary results for the embedding order.

There are still many open questions before a complete characterization of density or thinness preservation by morphisms or inverse morphisms can be achieved. The case of different alphabet sizes as well as the case of relations containing the embedding order would, for instance, need to be resolved. The pattern suggested by the results of this paper looks promising.

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