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Towards a neighborhood simplification of tile systems: From Moore to quasi-linear dependencies

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Abstract Self-assembly is the process by which objects aggregate independently and form complex structures. One of the theoretical frameworks in which the process of self-assembly can be embedded and formally studied is that of tile systems. A Wang tile is a square unit, with glues on its edges, attaching to other tiles which have matching glues, and forming larger and larger structures. In this paper we concentrate over two basic, but essential, self-assembling structures done by Wang tiles. The first one, called ribbon, is a non-self-crossing wire-like structure, in which successive tiles are adjacent along an edge, and where tiles are glued to their predecessor and successor by use of matching glues. The second one, called zipper, is a similar contiguous structure, only that here, all touching tiles must have matching glues on their abutting edges, independently of their position in the structure. In case of Wang tiles, it has been shown that these two structures are equivalent. Here we generalize this result for the case when the tiles have eight glues, four on their edges and four on their corners. Thus we show that an eight neighborhood dependency, namely the Moore neighborhood, can be simulated by a quasi-linear dependency.

Keywords Wang tiles · Self-assembly · Neighborhoods

1 Introduction

Nano-technology is a new emergent field of research and development of the last couple of decades. It has numerous applications in many areas such as medicine, electronics, material sciences, and technology, where there is a growing need of miniaturization. One of the main streams of research of this field is that of self-assembly. Here, a system is supposed to construct itself starting from its most basic components. The approach is

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highly attractive for nano-technology, as both the building blocks but also the entire system is supposed to be of a very small order, i.e., 10^{-9} m. Tile systems are a very well suited mathematical formalism able to capture and mimic the behaviors of self assemble systems.

Although tilings have appeared first as a form of art and architecture, they also have a long history as a mathematical concept. The first formal results on tiling by regular polygons appear already in 1619 in the *Harmonices Mundi* of Johannes Kepler, see Grünbaum and Shephard (1987). Introduced by Hao Wang in 1961 (Wang 1961, 1965), Wang tiles represent a new landmark in the history of tilings, shifting this field of science by introducing new formalisms and new perspectives.

A Wang tile is an oriented, i.e., un-rotatable, unit square with each edge covered by a specific "glue". The tiles are placed on the two-dimensional plane, and two adjacent tiles stick together if and only if they have the same glue on their abutting edges. In the last years, Wang tiles and some of their variants, e.g., the Tile Assembly Model (Rothemund 2001; Winfree 1998), have been widely used as models for self-assembly, many successful experimental results being thus obtained, see e.g., Barish et al. (2005) and Rothemund et al. (2004).

In Adleman et al. (2009), the authors investigate two basic (but essential) self-assembling structures done by Wang tiles. The first structure, called zipper, is a non-self-crossing sequence of tiles, in which successive tiles are adjacent along an edge, and all abutting edges of all tiles have matching glues. The second structure, called ribbon, is also a non-self-crossing sequence of tiles but here, only the tiles which are consecutive in the structure must have matching glues on their abutting edges. This second structure is a clear simplification of the first one. In zippers, the constituent tiles must satisfy the glue constraint imposed by four neighboring tiles. Contrary, in ribbons, the tiles must satisfy only a quasi-linear constraint, i.e., must have matching glues only with two of their adjacent neighbors. However, the authors show that the two structures are equivalent in the following way: for any set of tiles S_1 one can construct another one S_2 such that there is a one-to-one correspondence between zipper structures assembled using the S_1 tiles and ribbon structures assembled using S_2 tiles. That is, although it has less constrains, the ribbon structure can be used in order to replace and simulate the zipper structure.

Derived from geometrical tiles, Wang tiles have greatly simplified the topological aspect of tiling. Contrary to various geometrical tiles, a Wang tile can have at most four neighbors, placed directly to the North, East, South, or West. This may generate fundamental problems when modeling self assemble systems in which the constituent elements are interacting with more than only four surrounding components, placed particularly on the above directions. This paper addresses the problem of simulating complex neighborhoods by simpler ones. Thus, we define a new set of tiles, also in a square shape, but interacting with all the eight surrounding neighbors: the four ones on the horizontal and vertical directions, and the four ones on the diagonal directions. The interaction is done similarly by use of glues; each tile has eight glues, each associated either with an edge or with a corner. The tiles are placed on the two-dimensional plane, and two adjacent tiles stick together if and only if one of the following two situations happens: either the two tiles share a common edge, in which case they must have matching glues on their common edge, or they are in diagonal direction, in which case they must have matching glues on their common corner. In this framework, we redefine the zipper structure as follow. It is also a non-self-crossing sequence of tiles in which successive tiles are adjacent along an edge. However, a tile must have matching glues with all the surrounding tiles from the structure (at most eight), placed either on the horizontal and vertical direction, or on the diagonals directions. We show here that even if we use this more elaborated neighborhood, the classical ribbon structure composed of Wang tiles is still complex enough in order to replace and simulate the zipper structure. That is, we show that for any set S_1 of eight-glue tiles we can construct a set S_2 of Wang tiles, such that, up to some constrains, there is a one-to-one correspondence between zipper structures assembled using the S_1 tiles and ribbon structures assembled using S_2 tiles. Thus, we prove that an eight neighborhood dependence can be still simulated by a quasi-linear one, making this result notable also from a topological point of view.

The paper is organized as follows. The following section presents basic definitions regarding Wang tiles. In Sect. 3 we introduce our eight-glue tiles, and generalize the notion of zipper. In order to be able to simulate the new zipper structures by Wang tile ribbons we have to define a new type of tiles, using a different shape. These tiles, called Moore tiles, are introduced in Sect. 4, where we also define the notion of Moore tile zipper. In Sect. 5 we show how to simulate a Moore tile zipper by a Wang tile ribbon, such that in Sect. 6 we finish our construction by finally showing how to simulate an eight-glue zipper by a Wang tile ribbon. Our constructions are based on some motif designs which use geometry in order to simulate the glues of the edges and of the corners of the tiles.

2 Preliminaries

A *Wang tile* (or simply a *tile*) is an oriented unit square, i.e., a square which cannot be rotated or reflected, whose edges are labeled by symbols from a finite alphabet X, called the *set of glues*. Thus, each tile t is uniquely determined by the four glues of its North, East, South and West edges:

$$t = (t_N, t_E, t_S, t_W) \in X^4.$$

The tiles are placed on the nodes of a square lattice, and their positions are indexed by pairs of integers, $(i,j) \in \mathbb{Z}^2$. We say that two tiles t and t' *stick* on the North-South direction if and only if $t_N = t'_S$, that is, if both tiles have the same glue on their abutting edges, i.e., the North edge of t and the South edge of t'. Similarly, we can define the sticking property on the East–West, South–North and West–East directions.

We define the set of directions $\mathcal{D} = \{N, E, S, W\}$ as a collection of functions on $\mathbb{Z}^2$, as follows:

- $N(x, y) = (x, y + 1)$,
- $E(x, y) = (x + 1, y)$,
- $S(x, y) = (x, y - 1)$, and
- $W(x, y) = (x - 1, y)$.

We say that two positions $(x,y), (x',y') \in \mathbb{Z}^2$ are *adjacent* iff $(x',y') \in \{N(x,y), E(x,y), S(x,y), W(x,y)\}$. Moreover, we say that (x, y) *abuts* (x',y') to the North (resp. to the East, South, or West) iff $(x',y') = N(x,y)$ (resp. $(x',y') = E(x,y), (x',y') = S(x,y)$, or $(x',y') = W(x,y)$).

A *tile system T* is a finite collection of tiles. A *total T-tiling of the plane* is a mapping $\mathcal{T} : \mathbb{Z}^2 \to T$ which assigns to every position from $\mathbb{Z}^2$ a tile from T. We say that the tiling $\mathcal{T}$ is *valid on a position* $(i, j) \in \mathbb{Z}^2$, if the tile on that position, denoted as $t(i, j)$, sticks to all its adjacent neighbors, that is, the tiles $t(i, j+1), t(i+1, j), t(i, j-1)$, and $t(i-1, j)$ respectively. We say that $\mathcal{T}$ is valid, if it is valid on all positions $(i, j) \in \mathbb{Z}^2$. Note that in the case of total tilings we impose the existence of a tile on each position of the plane.

A *partial T-tiling* on a given domain $D \subseteq \mathbb{Z}^2$, is a mapping $\mathcal{T}_D : D \to T$, which assigns a tile from T to every position from D. We say that $\mathcal{T}_D$ is *valid* if for any tile within the domain there are no mismatches between the glues of that tile and the glues of its existing neighbors. More formally, for all $(i, j) \in D$

- if $(i, j+1) \in D$ then $t(i, j)_N = t(i, j+1)_S$;
- if $(i+1, j) \in D$ then $t(i, j)_E = t(i+1, j)_W$;
- if $(i, j-1) \in D$ then $t(i, j)_S = t(i, j-1)_N$;
- if $(i-1, j) \in D$ then $t(i, j)_W = t(i-1, j)_E$.

If no confusion can arise, we refer to valid tilings (either total or partial ones) simply as tilings.

A *path P* is a succession of adjacent positions in the plane. Formally, it is a mapping $P : I \to \mathbb{Z}^2$ where I is a set of consecutive integers, and for all $i, i+1 \in I, P(i)$ and $P(i+1)$ are adjacent.

A *T-tilled path* is a contiguous succession of tiles. Formally, it is a pair (P, r) where $P : I \to \mathbb{Z}^2$ is a path and $r : range(P) \to T$ is a mapping assigning tiles to all positions of the path.

A T-tilled path (P, r) is a *T-ribbon* if P is injective, i.e., the path is not self-crossing, and any tile in the path (except for the one on the last position, if any), must agree with its successor on the glues of their corresponding abutting edges, see Fig. 1a). That is, for all $i, i+1 \in dom(P)$ such that $P(i)$ abuts $P(i+1)$ to the *a)* North, *b)* East, *c)* South, *d)* West, we have that

- *a)* $r(P(i))_N = r(P(i+1))_S$,
- *b)* $r(P(i))_E = r(P(i+1))_W$,
- *c)* $r(P(i))_S = r(P(i+1))_N$,
- *d)* $r(P(i))_W = r(P(i+1))_E$.

Thus, in the case of T-ribbons any two adjacent tiles must agree on the glues of their corresponding abutting edges only if they are in consecutive positions inside the path. That is, each tile from the ribbon (except from the first and the last ones) is required to agree on its glues only with two of its neighbors, the predecessor and successor, and not with all four. Hence, in this case we say that we use a *quasi-liner neighborhood* (or a *partial von Neumann neighborhood*).

A *T-zipper* is a special type of a T-ribbon (P, r) where whenever two tiles are adjacent, regardless of whether they are in consecutive positions in the path, the two tiles must have the same glue of their abutting edges, see Fig. 1b). Formally, whenever for some $i, j \in dom(P), P(i)$ abuts $P(j)$ to the *a)* North, *b)* East, *c)* South, *d)* West, we have that

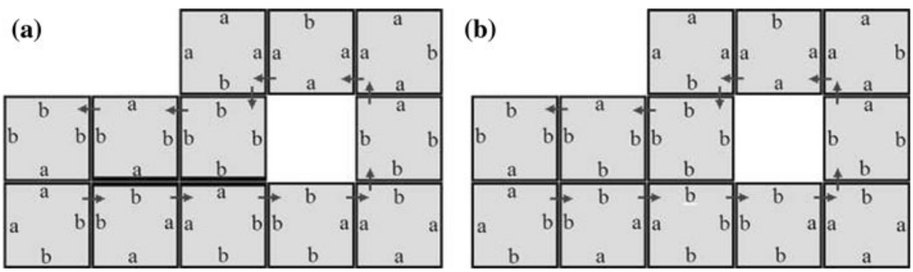


Fig. 1 **a** A Wang tile ribbon (which is not a zipper, see the underlined edges), **b** a Wang tile zipper

- *a)* $r(P(i))_N = r(P(j))_S$,
- *b)* $r(P(i))_E = r(P(j))_W$,
- *c)* $r(P(i))_S = r(P(j))_N$,
- *d)* $r(P(i))_W = r(P(j))_E$.

Given a ribbon (resp. a zipper) (P, r), we refer to P as the *underlying path* of the ribbon (resp. of the zipper).

In Adleman et al. (2009) it was shown that given a *directed* tile system T, i.e., a tile system where each tile has associated also a direction, one can construct a new tile system T' such that any directed T-zipper can be simulated by a T'-ribbon. That is, the authors showed that there exists an one-to-one correspondence between directed T-zippers and T'-ribbons. The paper proposes a motif construction, in which the authors use geometry in order to represent glues. An immediate consequence of this construction is that a von Neumann neighborhood can be simulated using a partial von Neumann neighborhood, i.e., a tile must agree on its glues only with its predecessor and its successor, and not with all the four neighbors. We extend this study and prove that even if we consider the *Moore neighborhood*, in which case each tile has as neighbors all the eight surrounding tiles, it is still possible to obtain a similar result. Thus, we show that given a tile system T which uses the Moore neighborhood, we can construct a tile system T' using the classical von Neumann neighborhood, such that any T-zipper can be simulated by a T'-ribbon. That is, we show that the eight-neighbor dependency imposed by the Moore neighborhood can be simulated by a quasi-linear dependency.

3 Tile systems using the Moore neighborhood

We say that a tile system uses the *Moore neighborhood* if each tile interacts with all its eight surrounding tiles. Thus, to each tile we associate eight glues, one for each of the following directions: North, Northeast, East, Southeast, South, Southwest, West, and Northwest, respectively. Then, two tiles which are adjacent on some of the eight possible directions, must agree on the corresponding glues. Just as before, the tiles are represented as oriented unit squares however, now we label with glues also the corners of these squares, see Fig. 2.

Formally, given a finite set of glues X, a tile system $T \subseteq X^8$ using the Moore neighborhood is a finite collection of tiles, where each tile $t \in T$ has eight glues,

$$t = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW}),$$

each glue corresponding to exactly one possible neighbor situated on the eight surrounding positions.

In addition to the four direction functions from the previous section we define the functions $NE, SE, SW, NW : \mathbb{Z}^2 \to \mathbb{Z}^2$, by $NE(x,y) = (x+1, y+1), SE(x,y) = (x-1, y+1), SW(x,y) = (x-1, y-1)$, and $NW(x,y) = (x-1, y+1)$. Also, we extend in the natural way the notion of adjacency to incorporate the new four direction functions, and

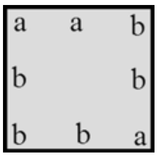


Fig. 2 A tile $t \in X^8$ using the Moore neighborhood

moreover, we also say that (x, y) abuts (x', y') to the Northeast (resp. to the Southeast, Southwest, Northwest) iff $(x', y') = NW(x, y)$ (resp. $(x', y') = SE(x, y)$, $(x', y') = SW(x, y)$, $(x', y') = NW(x, y)$).

The notions of total and partial tilings are generalized in the natural way. The tiles are still placed on the nodes of a square lattice, but each tile has to agree on its glues with the corresponding glues of all existing neighbors, among the eight surrounding tiles.

The notion of path (resp. tiled path) remains unchanged independently of whether we use the von Neumann neighborhood, or the Moore neighborhood. That is, a path P (resp. a tiled path) is a mapping $P : I \to \mathbb{Z}^2$ (resp. a pair (P, r)) where the element $P(i+1)$ can be placed only to the North, East, South or West of $P(i)$. Thus, also the definition of a T-ribbon, (P, r), remains unchanged, even if we require the tile system T to use the Moore neighborhood. Hence, for a T-ribbon (P, r), the underlying path P is not self-crossing, and each tile from the ribbon must agree on its glues only with its predecessor and its successor. However, in the case of T-zippers, we require any two adjacent tiles to agree on their corresponding common glues, including the cases when the two tiles are adjacent on a Northeast, Southeast, Southwest, or Northwest direction.

Our goal is to develop a method by which, given a tile system T using the Moore neighborhood, to construct a Wang tile system T', i.e., using the von Neumann neighborhood, such that any T-zipper could be simulated by a T'-ribbon. We show that up to some constraints, see Theorem 6.1, we can construct a one-to-one correspondence between T-zippers and T'-ribbons. As an intermediate step, we introduce a new type of tiles, called *Moore tiles*, which present some advantages in the way they directly communicate with their neighbors.

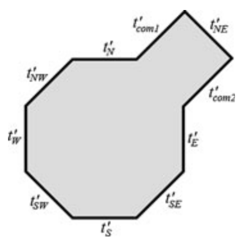
4 The Moore tiles

Given a set of glues X and a tile system $T \subseteq X^8$ of square tiles using the Moore neighborhood, we introduce a new tile system T' with tiles of an un-regular shape, which is used to simulate the tiles from T. We call these tiles *Moore tiles*. The shape of each tile is that of a regular octagon, whose Northeast edge is replaced by a square, see Fig. 3. When placed on the nodes of a square lattice, this shape can be used in order to perfectly tile the plane with no overlaps, as it is derived from one of the 11 Archimedean Tilings done by regular polygons, see Grünbaum and Shephard (1987).

A Moore tile $t' \in T'$ has ten edges: one edge corresponding to each of the eight directions, and two extra edges used for communication. All ten edges are labeled by glues from X, the only restriction being that, by definition, the two edges used for communication must have identical glues. Thus,

$$t' = (t'_N, t'_{com1}, t'_{NE}, t'_{com2}, t'_E, t'_{SE}, t'_S, t'_{SW}, t'_W, t'_{NW}) \in X^{10}, \text{ where } t'_{com1} = t'_{com2}.$$

Fig. 3 A Moore tile t'



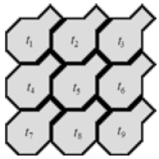


Fig. 4 A 3 × 3 lattice of Moore tiles

When placed on the nodes of a square lattice, each Moore tile is in direct edge-to-edge contact with six surrounding tiles, see Fig. 4. Thus, in this case, a tile has six neighbors (that is six surrounding tiles with common abutting edges), and moreover, two adjacent tiles can have either one or two common edges.

The notions of total or partial tilings, tiled paths, and zippers are generalized in the natural way. Thus, for a tiling $\mathcal{T}$ to be valid on some position $(i, j) \in \mathbb{Z}^2$, we require the tile $t(i, j)$ placed on that position to agree on the glues of its abutting edges with all the existing surrounding tiles. In particular, in the case of total tilings, the tile $t(i, j)$ must agree on the glues of the corresponding edges with all the tiles $t(i, j+1), t(i+1, j+1), t(i+1, j)$, $t(i, j-1), t(i-1, j-1)$, and $t(i-1, j)$, respectively. Also, in the case of tiled paths and zippers (P, r), for any $i, i+1 \in dom(P)$, we have that $P(i+1) = D(P(i))$, where $D \in \{N, E, S, W\}$.

Observation 1 Note that due to the communicating edges, a Moore tile agrees on the glues of its edges with more that its six immediate neighbors. For instance, consider the nine Moore tiles $t^1, t^2, \ldots, t^9 \in X^{10}$ placed on the nodes of a 3 × 3 square lattice from Fig. 4. If this partial tiling is valid, we can conclude the following:

$$t^5_N = t^2_S; \qquad t^5_{NE} = t^3_{SW}; \qquad t^5_E = t^6_W; \qquad t^5_S = t^8_N; \qquad t^5_{SW} = t^7_{NE}; \qquad t^5_W = t^4_E.$$

Moreover, since for all tiles t^i, $1 \le i \le 9$, by definition $t^i_{com1} = t^i_{com2}$, we also have that

$$t^5_{SE} = t^8_{com1} = t^8_{com2} = t^9_{NW} \text{ and } t^5_{NW} = t^4_{com2} = t^4_{com1} = t^1_{SE}.$$

Thus, we can say that the Moore tile t^5 "agrees" with all the eight "surrounding" tiles $t^1, \ldots, t^4, t^6, \ldots t^9$, although, two of these "neighbors", that is the tiles on the Northwest and the Southeast direction, are not actually touching the tile t^5.

For a given tile system $T \subseteq X^8$ using the Moore neighborhood, we associate a Moore tile system $T' \subseteq X^{10}$ as follows. If $X = \{x_1, x_2, \ldots, x_n\}$ is the set of glues then, for each tile $t \in T, t = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW})$ we introduce n Moore tiles $t^1, \ldots, t^n \in T'$, with

$$t^i = (t^i_N, t^i_{com1}, t^i_{NE}, t^i_{com2}, t^i_E, t^i_{SE}, t^i_S, t^i_{SW}, t^i_W, t^i_{NW}) \in X^{10}$$

such that for all $1 \le i \le n$ and all $edge \in \{N, NE, E, SE, S, SW, S, NW\}$ we have

$$t^i_{edge} = t_{edge} \quad \text{and} \quad t^i_{com1} = t^i_{com2} = x_i.$$

That is, each tile from T is replaced by n tiles in T', where these Moore tiles differ from each other only on the glues from the edges $com1$ and $com2$.

Theorem 4.1 *Let $T \subseteq X^8$ be a tile system using the Moore neighborhood, and let T' be its associated Moore tile system. Then, there exists a total tiling of the plane using the tiles from T if and only if there exists a total tiling of the plane using the tiles from T'. In particular, there exists a one-to-one correspondence between the total tilings of the plane obtained with the two tile systems.*

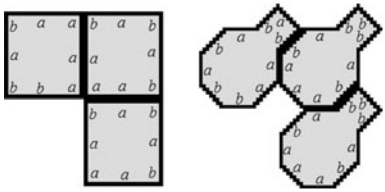


Fig. 5 Two corresponding partial tilings, one with square tiles (that is not valid) and the other with Moore tiles (that is valid)

Proof Let $\mathcal{T} : \mathbb{Z}^2 \to T$ be a total T-tiling of the plane. Then, we define the tiling $\mathcal{T}' : \mathbb{Z}^2 \to T'$ as follows. For each $(i, j) \in \mathbb{Z}^2$, if $t(i, j) = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW})$ and $t(i+1, j) = (\tau_N, \tau_{NE}, \tau_E, \tau_{SE}, \tau_S, \tau_{SW}, \tau_W, \tau_{NW})$ then we define $t'(i, j) = (t'_N, t'_{com1}, t'_{NE}, t'_{com2}, t'_E, t'_{SE}, t'_S, t'_{SW}, t'_W, t'_{NW})$ such that for all $edge \in \{N, NE, E, SE, S, SW, S, NW\}$ we have $t'_{edge} = t_{edge}$ and $t'_{com1} = t'_{com2} = \tau_{NW}$. Clearly, the new obtained tiling $\mathcal{T}'$ is valid in all positions $(i, j) \in \mathbb{Z}^2$. Moreover, since for each $(i, j) \in \mathbb{Z}^2$, the tile $t'(i, j)$ is uniquely determined by the tiles $t(i, j)$ and $t(i + 1, j)$, we conclude that to each total T-tiling we associate a unique total T'-tiling.

For the converse, starting with a total T'-tiling, we can obtain a total T-tiling of the plane by simply replacing each tile $t'(i, j) = (t'_N, t'_{com1}, t'_{NE}, t'_{com2}, t'_E, t'_{SE}, t'_S, t'_{SW}, t'_W, t'_{NW})$ with the tile $t(i, j) = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW})$ such that for all $edge \in \{N, NE, E, SE, S, SW, S, NW\}, t'_{edge} = t_{edge}$. The new obtained tiling is valid due to Observation 1. Moreover, by this construction, it is obvious that to each total T'-tiling we associate a unique total T-tiling. □

The advantage of using Moore tiles instead of square ones, is given by the fact that now, the glues of a tile are associated only with its edges, and not with its corners. Thus, verifying whether a tiling is valid on a particular position is reduced to verifying whether the tile on that position agrees on the glues of its edges with all the surrounding tiles. Moreover, although in the initial tile system the neighborhood of a tile is larger, i.e., it contains eight tiles, by using Moore tiles, we reduce it to the six surrounding ones.

Observation 2 The previous theorem shows that a total tiling of the plane generated by tiles using the Moore neighborhood can be simulated by a tiling using Moore tiles. However, this does not hold anymore when we consider partial tilings. In particular, there exist valid partial tilings using Moore tiles which can not be translated into valid partial tilings using square tiles, see for example Fig. 5. In these cases, although the partial tilings are valid in the case of Moore tiles, i.e., all Moore tiles agree on the glues of their edges with all the surrounding tiles, the corresponding tilings done by square tiles may not be valid anymore due to some mismatches on the Northwest–Southeast diagonal direction. In Sect. 6 we overcome this problem by introducing modified ribbon motifs, such as those reported in Fig. 10.

5 The ribbon construction

We are now ready to go to the second step of our construction and simulate Moore tile zippers with ribbons. Recall that a zipper is a more complex construction than a ribbon. Indeed, in a ribbon, only consecutive tiles are required to agree on the glues of their common abutting edges. However, in a zipper, even non-consecutive neighboring tiles are required to have matching glues on their corresponding abutting edges. The idea of the construction is very similar to the one from Adleman et al. (2009).

Given a Moore tile system $T' \subseteq X^{10}$, we construct an associated Wang tile system $T'' \subseteq X^4$, i.e., using the von Neumann neighborhood, as follows. For each Moore tile $t' \in T'$ we construct twelve T''-motifs.

A *motif* is a finite T''-ribbon (P, r) whose underling path outlines the contours of a Moore tile. Moreover, on the sides of these ribbon motifs we place matching dents and bumps, see Fig. 6, in order to simulate the glues of the Moore tile edges.

Observation 3 In order to be able to outline the contour of the Moore tiles by a contiguous sequence of small squares, we had to slightly modify the shape of these tiles, as illustrated in Fig. 7. From now on, when we talk about Moore tiles, we refer to these modified tiles. Due to the high level of formalism needed in order to formally prove that this construction can indeed be used in order to tile the plane simply by translating it horizontally and vertically, see e.g., Fig. 8, we omit this proof here. For example, one way to prove that we obtain a non-overlapping tiling, is to formally define this shape by giving the exact position of each small square from the contour, and show that whenever two such shapes are adjacent, there are no gaps between them and there are no overlaps either.

In order to underline the difference between Moore tiles and their associated motifs, we fix the following notations. When considering Moore tiles we talk about edges of the tile, that is the North edge, the com1 edge, and so on. When considering motifs, we talk about *sides* of the motifs, that is, the North side, the com1 side, and so on.

The first tile of a motif is the *input tile*, and it is placed midway along one of the North, East, South or West sides of the motif. This side of the motif is called the *input side*. The last tile of the motif is the *output tile*, and it is placed midway along one of the North, East, South or West sides of the motif, other than the input side. This side of the motif is called

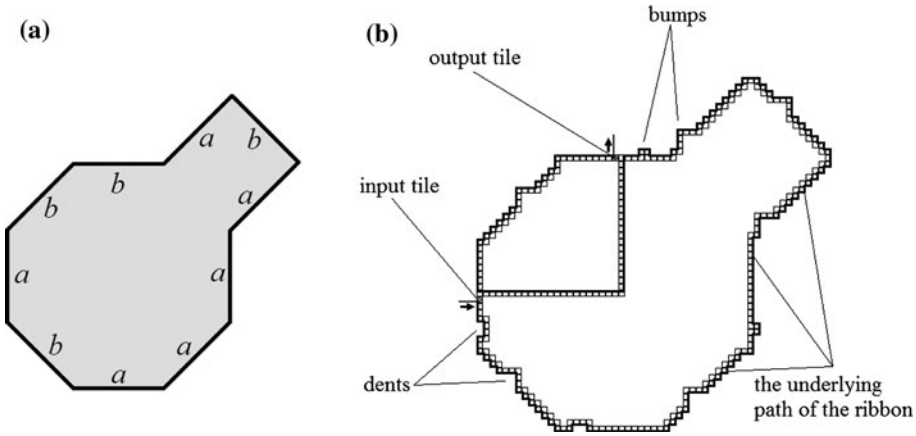
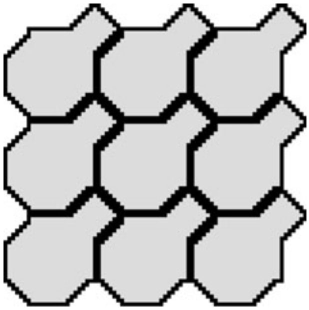


Fig. 6 **a** A Moore tile t', **b** One of the ribbon motifs associated to t'

Fig. 7 A modified Moore tile

Fig. 8 A 3 × 3 lattice of modified Moore tiles



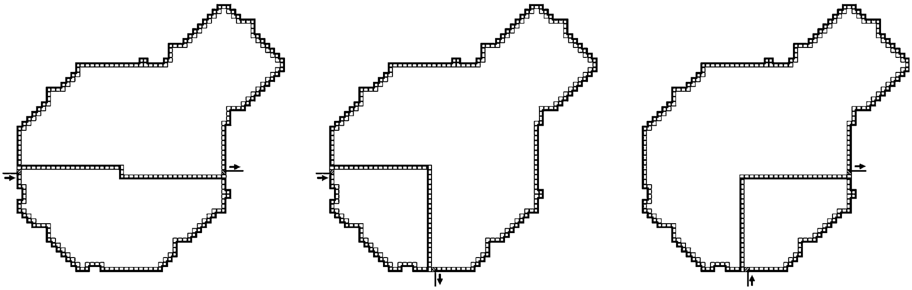


Fig. 9 Three of the ribbon motifs associated to the Moore tile from Fig. 6

the *output side*. Given a Moore tile $t' \in T'$, we associate to it twelve T''-ribbon motifs, each obtained by selecting all possible distinct pairs for the input/output sides. These motifs are called *the variants* of t'. In Fig. 6b we already presented one of the variants associated to the Moore tile from Fig. 6a; in Fig. 9 we describe three more variants of the same tile.

The bumps and dents from the sides of the motifs are used in order to simulate the glues of the edges of the Moore tile t', and are identical in all twelve variants. The bumps are placed on the North, com1, Northeast, East and Northwest sides, while the dents are placed on the com2, Southeast, South, Southwest and West sides, see Fig. 6. To each glue we associate a unique position along each side of a motif, such that if the glues of two adjacent tiles from the T'-zipper are matching, then the bumps and the dents from the corresponding sides of the two associated motifs are placed such that they fit each other. That is if, for example, the Moore tile t'_1 sticks on the East and com2 edges with the Moore tile t'_2, then the bump on the East side of any of the t'_1 variants fits exactly with the dent on the West side of any of the t'_2 variants. Also, the dent on the com2 side of any of the t'_1 variants fits exactly with the bump on the Northwest side of any of the t'_2 variants. If however the Moore tile t'_1 does not stick to the Moore tile t'_2 on one abutting edge, say for example on the East edge, then whenever we place any of the t'_1 and t'_2 variants side-by-side, the bump from the East side of the t'_1 variant overlaps with the West side of the t'_2 variant. Since in ribbons we do not allow overlaps, we obtain that if the Moore tile t'_1 does not have on its East and com2 edges the same glues as the tile t'_2 has on its West and Northwest edges, respectively, then no T''-ribbon can have a t'_2 variant motif directly to the East of a t'_1 variant motif.

We consider next the glues of the ribbon-tiles, i.e., the tiles from T''. The following encoding is very similar to the one used in Adleman et al. (2009). The input and output ribbon-tiles of each variant motif have a special labeling on those edges connecting the motif with the preceding, respectively the succeeding motif. The glues of all the other tiles are designed so that each tile can be placed in exactly one variant motif, and moreover, it occurs exactly once in that variant. In order to achieve this, for each position of any of the variant motifs (except the first and the last), the tile on that position has two unique glues on the two edges connecting it with the preceding and respectively succeeding tiles from the ribbon. The two remaining edges of each tile, i.e., the edges which are not in contact with the preceding or the succeeding tile from the ribbon, have to be labeled such that no unwanted sticking can occur between: (i) tiles of the same variant motif, and (ii) tiles from adjacent motifs. For that, we create two totaly new glues, called *null*1 and *null*2 and, unless they were already labeled by the above process, the North and East edges of a tile are labeled with *null*1, while the South and West edges are labeled with *null*2.

We consider now the labeling of the input and output ribbon-tiles (of the variant motifs). The glues of the edges connecting these tiles to the rest of the motif are chosen as previously, i.e., in a unique manner such that to ensure that the tile can be placed in exactly one variant motif, and in one position only. For those edges connecting the motif with the preceding and respectively the succeeding ribbon-motifs we create four new glues *North-to-South*, *East-to-West*, *South-to-North* and *West-to-East*, each encoding a specific motif-to-motif transition. Thus depending on which of the sides of a motif is the input and respectively the output one, we proceed as follows:

- If the output side of the motif is North (resp. East, South, West), then the output tile from that ribbon motif has its North (resp. East, South, West) edge labeled by the glue *North-to-South* (resp. *East-to-West*, *South-to-North*, *West-to-East*).
- If the input side of the motif is North (resp. East, South, West), then the input tile from that ribbon motif has its North (resp. East, South, West) edge labeled by the glue *South-to-North* (resp. *West-to-East*, *North-to-South*, *East-to-West*).

For the remaining two edges of the input and output tiles, we apply the same *null*1 and *null*2 labeling technique as in the case of all the other ribbon-tiles. As a consequence, we ensure that all variant motifs are connected to each other exactly as in the case of the Moore tiles from the zipper they simulate. We say that a T''-ribbon is *complete* if either it is bi-infinite or, in the case that it has a first or a last tile, or both, then this must be an input and respectively an output tile.

To conclude, we use this construction for proving that any T'-zipper can be simulated by a complete T''-ribbon. In particular, we show that to any zipper we can associate a ribbon construction whose underling path outlines the contours of the Moore tiles, with some extra bumps and dents simulating the glues of the edges.

Theorem 5.1 *Let $T' \subseteq X^{10}$ be a Moore tile system, and let T'' be its associated Wang tile system. Then, there exists a one-to-one correspondence between T'-zippers and complete T''-ribbons, up to replacing the first and the last of the ribbon motifs (if they exist) with any other variants of those motifs, but with the same output and respectively input sides.*

Proof Let (P, r) be a T'-zipper. We replace each Moore tile from the zipper with its associated ribbon motif whose input and output sides correspond to the position of the previous and respectively next tile from the zipper. Thus, we obtain a contiguous T'' tiled path, where each ribbon motif is linked to the previous and to the next ribbon motif. Moreover, since all the Moore tiles in (P, r) have matching glues on all abutting edges, there are no overlaps between various bumps and dents of the T''-ribbon motifs. Thus, the entire contiguous structure is a T''-ribbon. Also, note that the first (resp. the last) Moore tile from the zipper, if it exists, can be replaced with any of the three corresponding variant motifs of that tile which have the same output (resp. the same input) side.

Let us suppose now that there exists a complete T''-ribbon. From the way we constructed the tiles of T'' we can conclude that this ribbon consists of a succession of ribbon motifs. Moreover, since there are no overlaps between any adjacent motifs, we conclude that all the bumps and dents are matching. Thus, by replacing all the ribbon-motifs with the corresponding Moore tiles, we obtain a T'-zipper. □

6 From arbitrary zipper structures to ribbons

By putting together Theorem 4.1 and Theorem 5.1 we can now conclude the following. For every tile system $T \subseteq X^8$ of square tiles and using the Moore neighborhood, we can construct a Wang tile system T'' (i.e., consisting also of square tiles but using the von Neumann neighborhood) such that there is a one-to-one correspondence between T-zippers covering the entire plane and bi-infinite T''-ribbons. Thus, we can say that every T-zipper covering the entire plane can be simulated by a T''-ribbon. However, due to Observation 2 we cannot obtain a similar conclusion when we consider T-zippers describing only a partial tiling of the plane.

Indeed, the problem appears when we convert a partial tiling done by Moore tiles into a partial tiling done by square tiles using the Moore neighborhood, see Observation 2. More precisely, let us consider two Moore tiles $t'_1, t'_2 \in X^{10}$, one Northwest to the other. That is, let t'_1 and t'_2 be two tiles placed on positions (i, j) and $(i-1, j+1)$, respectively. Then, the "information" regarding the glue t'_{1NW} placed on the Northwest edge of t'_1 is transmitted from the tile t'_1 to t'_2 only if there exists an intermediate tile $t'_3 \in X^{10}$ in between, i.e., on position $(i-1, j)$. If this third tile is missing, i.e., the position $(i-1, j)$ is not included in the domain of the partial tiling, then we cannot enforce the glue of the Northwest edge of t'_2, i.e., t'_{1NW}, to be identical with the glue of the Southeast edge of t'_2, i.e., t'_{2SE}.

In order to facilitate the "transfer of information" on the Northwest-Southeast diagonal, independently of whether there exists an intermediate tile, we introduce now some modified ribbon motifs for the variants associated to a Moore tile, see Fig. 10. In the new ribbon motifs we enlarge the sizes of the bumps and dents from the Northwest, Southeast, com1, and com2 sides as follows.

On the Northwest side of each motif, we elongate the existing bump as illustrated in Fig. 10. The new bump is called *spike* and it is long enough to "cross" the com1 and com2

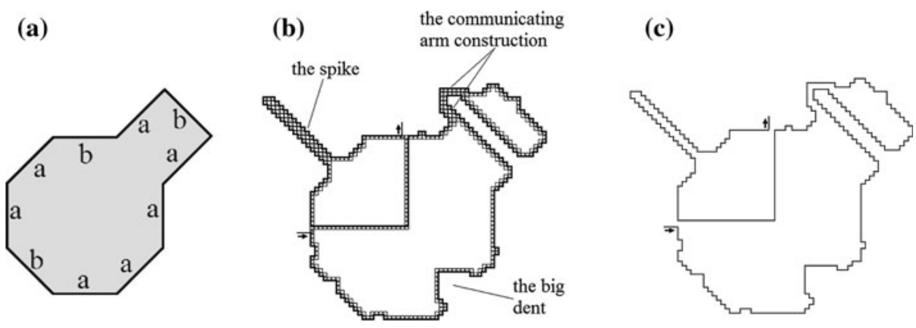


Fig. 10 **a** The Moore tile, **b** One of its modified ribbon motifs, **c** The underling path of the ribbon

sides of the ribbon motif placed to the West (if such a ribbon motif exists there), and the Southeast side of the ribbon motif placed to the Northwest (similarly, if such a ribbon motif exists there). Although we said that this spike "crosses" the three sides of the previous motifs (assuming these motifs exist), in fact these sides are modified accordingly, so that the spike does not overlap with them if and only if the modified bumps and dents from these sides correspond to the same glue.

On the com1 and com2 sides of each motif, we enlarge the existing bump and dent, see Fig. 10, such that if there exists another ribbon motif to the East, then its spike would be perfectly surrounded by the modified dent and bump only when the positions of the bump, dent, and spike correspond to the same glue. We call these matching bump and dent the *communicating arm construction*. Note that in the case of a Moore tile zipper, whenever a tile t_2 is placed to the East of a tile t_1, then the glues of the com1 and com2 edges of t_1 are the same as the glue of the Northwest edge of t_2.

Finally, on the Southeast side of each ribbon motif we also enlarge the existing dent, see Fig. 10, such that, if there exists a ribbon motif to the South, then its communicating arm construction would be perfectly surrounded by this enlarged dent if and only if they correspond to the same glue. We call this dent from the Southeast side of each ribbon motif the *big dent*. Moreover, this big dent is constructed such that even in the case when there is no ribbon motif to the South but there is one to the Southeast, the spike of the second motif will not overlap with the big dent of the first one if and only if the positions of the two constructions along the sides correspond to exactly the same glue.

Due to the above modifications, we have obtained the following. In the case two ribbon motifs are placed first Southeast of the other, then, independently whether there exists a third ribbon motif in between, i.e., to the South of the first one, there are no overlaps between the two motifs if and only if the big dent of the first one simulates exactly the same glue as the spike of the second. Thus, we can formulate the following result.

Theorem 6.1 *Let $T \subseteq X^8$ be a tile system using the Moore neighborhood. Then, we can construct a Wang tile system T'' such that any T-zipper can be simulated by a complete T''-ribbon. That is, there exists a one-to-one correspondence between T-zippers and complete T''-ribbons, up to replacing some of the ribbon motifs with other variants of the same motifs, or with other ribbon motifs which differ only on the com1 and com2 sides.*

Proof Let (P, r) be a T-zipper. Recall that to the tile system $T \subseteq X^8$ we can associate a Moore tile system $T' \subseteq X^{10}$, as presented in Sect. 4. Then, we create a T'-zipper (P, r') as follows. For each $(i, j) \in range(P)$ we replace the tile $t = r(i, j), t = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW})$ with a Moore tile $t' = r'(i, j), t' = (t'_N, t'_{com1}, t'_{NE}, t'_{com2}, t'_E, t'_{SE}, t'_S, t'_{SW}, t'_W, t'_{NW})$ as follows:

(i) If $(i+1, j) \in range(P)$, then for all $edge \in \{N, NE, E, SE, S, SW, W, NW\}$ we have $t'_{edge} = t_{edge}$ and $t'_{com1} = t'_{com2} = r(i+1, j)_{NW}$.
(ii) If $(i, j+1) \in range(P)$, then for all $edge \in \{N, NE, E, SE, S, SW, W, NW\}$ we have $t'_{edge} = t_{edge}$ and $t'_{com1} = t'_{com2} = r(i, j+1)_{SE}$.
(iii) If $(i+1, j), (i, j+1) \notin range(P)$, then for all $edge \in \{N, NE, E, SE, S, SW, W, NW\}$ we have $t'_{edge} = t_{edge}$ and $t'_{com1} = t'_{com2} = x$, where $x \in X$ is any glue from X.

Note that if both $(i+1, j), (i, j+1) \in range(P)$, since (P, r) is a zipper, we have that $r(i+1, j)_{NW} = r(i, j+1)_{SE}$. Thus, the above process does not generate a contradiction. Clearly the new obtained tiled path is a T'-zipper since the path P is not self-crossing and on all abutting edges we have matching glues. However, note that by the above process we can obtain several T'-zippers since some of the tiles from the T-zipper can be replaced by several Moore tiles which differ only on the glues of the *com*1 and *com*2 edges, see the case *iii*) above. □

Next, we construct the Wang tile system T'' associated to T'. Similarly to the first part of the proof of Theorem 5.1, but using the modified ribbon motifs, we associate to the T'-zipper a complete T''-ribbon. Moreover, this ribbon is unique, up to replacing the first and the last of the ribbon motifs (if they exist) with any other variants of those motifs, but with the same output and respectively input sides. Thus, we obtain a transition from T-zippers to complete T''-ribbons.

In Fig. 11 we present an example of the previously described transition from zippers to ribbons. To the T-zipper illustrated in Fig. 11a, we associate the T'-zipper using Moore tiles from Fig. 11b. Note that this T'-zipper is not unique, as the *com*1 and *com*2 edges of the third Moore tile (and only of that tile) could be also labeled by the glue b. Finally, in Fig. 11c, we have the underling path of the T''-ribbon associated to the T'-zipper, and thus, associated also to the T-zipper (for a better visualization of the construction we included just the underling path and not the actual ribbon). Note that again we can have several variants for this T''-ribbon, depending of the input and output sides of the first and respectively last of the ribbon motifs.

For the converse implication, let us start now with a complete T''-ribbon. From the way we constructed the tiles in T'' we conclude that this T''-ribbon is a succession of ribbon motifs, associated to a non-self-crossing Moore tiled path. Moreover, since there are no overlaps between various bumps and dents of these motifs, we conclude that on all pairs of abutting edges of the Moore tiles we have matching glues, regardless of whether the Moore tiles are in consecutive positions inside the path. Also, since the spikes of the ribbon motifs do not overlap with the big dents, we conclude that whenever two Moore tiles are placed one Northwest to the other, the corresponding Northwest and Southeast edges have the same glue. Note that in the case of Moore tiles, these previous two edges are not actually abutting each other. Finally, we can switch to tiles from T simply by replacing each Moore tile $t' = (t'_N, t'_{com1}, t'_{NE}, t'_{com2}, t'_E, t'_{SE}, t'_S, t'_{SW}, t'_W, t'_{NW})$ with the tile $t = (t_N, t_{NE}, t_E, t_{SE}, t_S, t_{SW}, t_W, t_{NW})$, where for all $edge \in \{N, NE, E, SE, S, SW, W, NW\}$, $t_{edge} = t'_{edge}$. Clearly, the

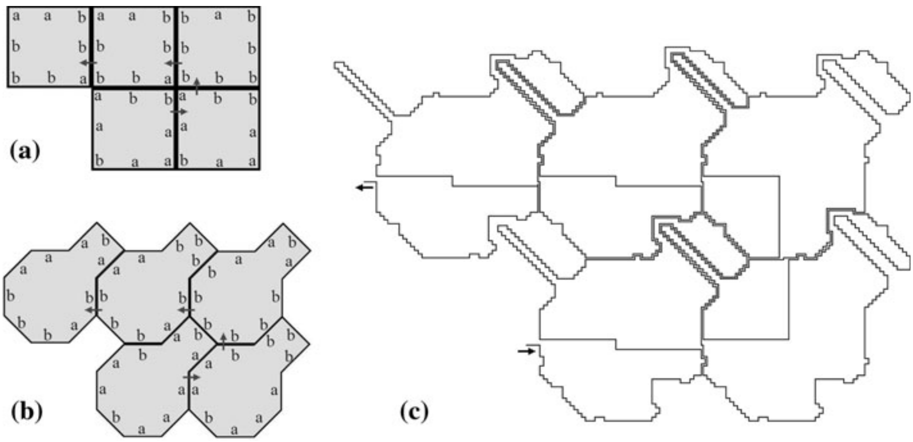


Fig. 11 **a** A T-zipper (the arrows on the tiles indicate the underling path), **b** One of its associated T'-zippers, **c** The underling path of one of the associated T''-ribbons

new obtained T-tiled path is a T-zipper. Moreover, this zipper can be derived from the initial T''-ribbon in a unique manner.

7 Conclusions

We have showed that even in the case when tiles have to match their glues with up to eight neighbors, a zipper construction can still be simulated by a wire-like structure, namely a ribbon. From a topological point of view, this result shows that an eight neighborhood dependency can be simulated by a quasi-linear dependency. This might be intriguing as it is known that some properties of one-dimensional systems are decidable, while the related properties in the two-dimensional space become undecidable. For example, we can consider here the reversibility of cellular automata, which is decidable in the one-dimensional space, see Amoroso and Patt (1972), and undecidable in two- and higher-dimensional spaces, see Kari (1994). However, note that in our case we use quasi-linear dependencies in which, even if a tile must agree only with its predecessor and with its successor, these two tiles can be placed on perpendicular directions. Thus, it still remains impossible to break the boundary between the one- and the two-dimensional spaces.

In Czeizler and Kari (2009) we have generalized (in some sense) our results from here for even more complex neighborhoods. Namely, we designed geometric tiles which simulate square tiles with multiple glues. Some of these glues correspond to edges abutting some local (touching) neighbors, while other glues correspond to virtual edges between the tile and some distant neighbors. By replacing the geometric tiles with appropriate ribbon motifs we can then simulate a zipper structure in which the tiles use the above complex neighborhoods by a ribbon structure having a quasi-linear neighborhood. However, the general problem of simulating any two-dimensional neighborhood by a quasi-linear one, proves to be very complex.

Acknowledgements The idea and the design of the Moore tiles from Sect. 4 are due to L. Adleman. Initially, they were called *pitcher-tiles* due to their shape. This research was supported by Natural Sciences and Engineering Council of Canada Discovery Grant R2824A01, Canada Research Chair Grant X3159A34, and Faculty of Science Grant R2824A03 to L.K., and Academy of Finland, project number 203667, and Foundation of Å bo akademi University for E.C.

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