Number Theory and Cryptography

Chapter 4

With Question/Answer Animations

Chapter Motivation

- *Number theory* is the part of mathematics devoted to the study of the integers and their properties.
- Key ideas in number theory include divisibility and the primality of integers.
- Representations of integers, including binary and hexadecimal representations, are part of number theory.
- Number theory has long been studied because of the beauty of its ideas, its accessibility, and its wealth of open questions.
- We'll use many ideas developed in Chapter 1 about proof methods and proof strategy in our exploration of number theory.
- Mathematicians have long considered number theory to be pure mathematics, but it has important applications to computer science and cryptography studied in Sections 4.5 and 4.6.

Chapter Summary

- Divisibility and Modular Arithmetic
- Integer Representations and Algorithms
- Primes and Greatest Common Divisors
- Solving Congruences
- Applications of Congruences
- Cryptography

Divisibility and Modular Arithmetic

Section 4.1

Section Summary

- Division
- Division Algorithm
- Modular Arithmetic

Division

Definition: If a and b are integers with $a \neq 0$, then *a divides b* if there exists an integer c such that $b = ac$.

- When a divides b we say that a is a *factor* or *divisor* of b and that b is a multiple of a.
- The notation $a \mid b$ denotes that a divides b.
- If $a \mid b$, then b/a is an integer.
- If a does not divide b, we write $a \nmid b$.

Example: Determine whether $3 \mid 7$

and whether $3 \mid 12$.

Properties of Divisibility

Theorem 1: Let a, b, and c be integers, where $a \neq 0$.

i. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;
ii. If $a \mid b$, then $a \mid bc$ for all integers c;
iii. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof: (i) Suppose $a \mid b$ and $a \mid c$, then it follows that there are integers s and t with $b = as$ and $c = at$. Hence,

$$b + c = as + at = a(s + t). \quad \text{Hence, } a \mid (b + c)$$

(parts (ii) and (iii)can be proven similarly) ◀

Corollary: If a, b, and c be integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ for any integers m and n.

Can you show how it follows easily from from (ii) and (i) of Theorem 1?

$$\underset{}{a} = d \cdot \overset{\text{quotient}}{(a \ \mathbf{div}\ d)} + \overset{\text{remainder}}{(a \ \mathbf{mod}\ d)}$$

Division Algorithm

- When an integer is divided by a positive integer, there is a quotient and a remainder.

Theorem ("Division Algorithm"): If a is an integer and d a positive integer, then there are unique integers q and r with $0 \le r < d$, such that $a = dq + r$ (*proved in Section* 5.2).

- a is called the *dividend*.
- d is called the *divisor*.
- q is called the *quotient*.
- r is called the *remainder*.

Definitions of Functions **div** and **mod**

$q = a \ \mathbf{div}\ d \leftarrow \left\lfloor \frac{a}{d} \right\rfloor$

$r = a \ \mathbf{mod}\ d$

Examples:

- What are the quotient and remainder when 101 is divided by 11?
  Solution: The quotient is $9 = 101 \ \mathbf{div}\ 11$ and the remainder is $2 = 101 \ \mathbf{mod}\ 11$.
- What are the quotient and remainder when 11 is divided by 3?
  Solution: The quotient is $3 = 11 \ \mathbf{div}\ 3$ and the remainder is $2 = 11 \ \mathbf{mod}\ 3$.
- What are the quotient and remainder when -11 is divided by 3?
  Solution: The quotient is $-4 = -11 \ \mathbf{div}\ 3$ and the remainder is $1 = -11 \ \mathbf{mod}\ 3$.

Congruence Relation

Definition: If a and b are integers and m is a positive integer, then a is *congruent* to b *modulo* m if m divides $a - b$.

- The notation $a \equiv b \pmod{m}$ says that a is congruent to b modulo m.
- We say that $a \equiv b \pmod{m}$ is a *congruence* and that m is its *modulus*.
- Two integers are congruent mod m if and only if they have the same remainder when divided by m. (Theorem 3 later)
- If a is not congruent to b modulo m, we write $a \not\equiv b \pmod{m}$

Example: Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Solution:

- $17 \equiv 5 \pmod{6}$ because 6 divides $17 - 5 = 12$.
- $24 \not\equiv 14 \pmod{6}$ since $24 - 14 = 10$ is not divisible by 6.

More on Congruences

Theorem 4: Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof:

- If $a \equiv b \pmod{m}$, then (by the definition of congruence) $m \mid a - b$. Hence, there is an integer k such that $a - b = km$ and equivalently $a = b + km$.
- Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, $m \mid a - b$ and $a \equiv b \pmod{m}$. ◄

The Relationship between (mod m) and **mod** m Notations

- The use of "mod" in $a \equiv b \pmod{m}$ is different from its use in $a = b \mathbin{\mathbf{mod}} m$.
  - $a \equiv b \pmod{m}$ - **mod** relates (two) sets of integers.
  - $a = b \mathbin{\mathbf{mod}} m$ - here **mod** denotes a function.

- The relationship/differences between these is clarifies below:

Theorem 3: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if

$$a \mathbin{\mathbf{mod}} m = b \mathbin{\mathbf{mod}} m.$$

(proof - *home exercise*)

Congruences of Sums and Products

Theorem 5: Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m} \quad \text{and} \quad ac \equiv bd \pmod{m}$$

Proof:

- Because $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, by Theorem 4 there are integers s and t with $b = a + sm$ and $d = c + tm$.
- Therefore,
  - $b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$ and
  - $b\,d = (a + sm)\,(c + tm) = ac + m(at + cs + stm)$.
- Hence, $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$.

◀

Example: Because $7 \equiv 2 \pmod{5}$ and $11 \equiv 1 \pmod{5}$, it follows from Theorem 5 that

$$18 = 7 + 11 \equiv 2 + 1 = 3 \pmod{5}$$
$$77 = 7 \cdot 11 \equiv 2 \cdot 1 = 2 \pmod{5}$$

Algebraic Manipulation of Congruences

- Multiplying both sides of a valid congruence by an integer preserves validity.

  If $a \equiv b \pmod{m}$ holds then $c \cdot a \equiv c \cdot b \pmod{m}$, where c is any integer, holds by Theorem 5 with $d = c$.

- Adding an integer to both sides of a valid congruence preserves validity.

  If $a \equiv b \pmod{m}$ holds then $c + a \equiv c + b \pmod{m}$, where c is any integer, holds by Theorem 5 with $d = c$.

- NOTE: dividing a congruence by an integer may not produce a valid congruence.

Example: The congruence $14 \equiv 8 \pmod{6}$ holds. Dividing both sides by 2 gives invalid congruence since $14/2 = 7$ and $8/2 = 4$, but $7 \not\equiv 4 \pmod{6}$. See Section 4.3 for conditions when division is ok.

Computing the **mod** *m* Function of Products and Sums

- We use the following corollary to Theorem 5 to compute the remainder of the product or sum of two integers when divided by m from the remainders when each is divided by m.

Corollary: Let m be a positive integer and let a and b be integers. Then

$$(a + b) \mathbf{\ mod\ } m = ((a \mathbf{\ mod\ } m) + (b \mathbf{\ mod\ } m)) \mathbf{\ mod\ } m$$

and

$$ab \mathbf{\ mod\ } m = ((a \mathbf{\ mod\ } m)(b \mathbf{\ mod\ } m)) \mathbf{\ mod\ } m.$$

(*proof in text*)

Arithmetic Modulo m

Definitions: Let $\mathbf{Z}_m = \{0, 1, \ldots., m-1\}$

be the set of nonnegative integers less than m. Assume $a, b \in \mathbf{Z}_m$.

- The operation $+_m$ is defined as $a +_m b = (a + b)$ **mod** m.
  This is *addition modulo m*.
- The operation $\cdot_m$ is defined as $a \cdot_m b = (a \cdot b)$ **mod** m.
  This is *multiplication modulo m*.
- Using these operations is said to be doing *arithmetic modulo m*.

Example: Find $7 +_{11} 9$ and $7 \cdot_{11} 9$.

Solution: Using the definitions above:

- $7 +_{11} 9 = (7 + 9) \bmod 11 = 16 \bmod 11 = 5$
- $7 \cdot_{11} 9 = (7 \cdot 9) \bmod 11 = 63 \bmod 11 = 8$

Arithmetic Modulo m

- The operations $+_m$ and $\cdot_m$ satisfy many of the same properties as ordinary addition and multiplication.

  - *Closure*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b$ and $a \cdot_m b$ belong to $\mathbf{Z}_m$.
  - *Associativity*: If a, b, and c belong to $\mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$.
  - *Commutativity*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b = b +_m a$ and $a \cdot_m b = b \cdot_m a$.
  - *Identity elements*: The elements 0 and 1 are identity elements for addition and multiplication modulo m, respectively.
    - If a belongs to $\mathbf{Z}_m$, then $a +_m 0 = a$ and $a \cdot_m 1 = a$.

continued →

Arithmetic Modulo *m*

- *Additive inverses*: If $a \neq 0$ belongs to $\mathbf{Z}_m$, then $m-a$ is the additive inverse of a modulo m and 0 is its own additive inverse.

  $a +_m (m-a) = 0$ and $0 +_m 0 = 0$

- *Distributivity*: If *a*, *b*, and *c* belong to $\mathbf{Z}_m$, then

  $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and
  $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$

- Multiplicatative inverses have not been included since they do not always exist. For example, there is no multiplicative inverse of 2 modulo 6, i.e.

  $2 \cdot_m a \neq 1$ for any $a \in \mathbf{Z}_6$

- (*optional*) Using the terminology of abstract algebra, $\mathbf{Z}_m$ with $+_m$ is a commutative group and $\mathbf{Z}_m$ with $+_m$ and $\cdot_m$ is a commutative ring.

Integer Representations and Algorithms

Section 4.2

Section Summary

- Integer Representations
  - Base *b* Expansions
  - Binary Expansions
  - Octal Expansions
  - Hexadecimal Expansions
- Base Conversion Algorithm
- Algorithms for Integer Operations

Representations of Integers

- In the modern world, we use *decimal,* or *base* 10, *notation* to represent integers. For example when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$.
- We can represent numbers using any base b, where b is a positive integer greater than 1.
- The bases $b = 2$ (*binary*), $b = 8$ (*octal*) , and $b = 16$ (*hexadecimal*) are important for computing and communications
- The ancient Mayans used base 20 and the ancient Babylonians used base 60.

Base *b* Representations

- We can use positive integer b greater than 1 as a base, because of this theorem:

  Theorem 1: Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

  $$n = a_k b^k + a_{k-1} b^{k-1} + \ldots. + a_1 b + a_0$$

  where k is a nonnegative integer, $a_0, a_1, \ldots. a_k$ are nonnegative integers less than b, and $a_k \neq 0$. The a_j, $j = 0,\ldots,k$ are called the base-b digits of the representation.

  (We will prove this using mathematical induction in Section 5.1.)

- The representation of n given in Theorem 1 is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} \ldots. a_1 a_0)_b$.

- We usually omit the subscript 10 for base 10 expansions.

Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Example: What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Solution:

$(1\ 0101\ 1111)_2 = 1\cdot2^8 + 0\cdot2^7 + 1\cdot2^6 + 0\cdot2^5 + 1\cdot2^4 + 1\cdot2^3 + 1\cdot2^2 + 1\cdot2^1 + 1\cdot2^0 = 351.$

Example: What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

Solution: $(11011)_2 = 1\cdot2^4 + 1\cdot2^3 + 0\cdot2^2 + 1\cdot2^1 + 1\cdot2^0 = 27.$

Octal Expansions

The octal expansion (base 8) uses the digits $\{0,1,2,3,4,5,6,7\}$.

Example: What is the decimal expansion of the number with octal expansion $(7016)_8$?

Solution: $7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8^1 + 6 \cdot 8^0 = 3598$

Example: What is the decimal expansion of the number with octal expansion $(111)_8$?

Solution: $1 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0 = 64 + 8 + 1 = 73$

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols. The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}. The letters A through F represent the decimal numbers 10 through 15.

Example: What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Solution:

$2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = 175627$

Example: What is the decimal expansion of the number with hexadecimal expansion $(1E5)_{16}$?

Solution: $1 \cdot 16^2 + 14 \cdot 16^1 + 5 \cdot 16^0 = 256 + 224 + 5 = 485$

Base Conversion

To construct the base b expansion of an integer n (in base 10):

- Divide n by b to obtain a quotient and remainder.

  $n = bq_0 + a_0 \quad 0 \leq a_0 \leq b$

- The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b.

  $q_0 = bq_1 + a_1 \quad 0 \leq a_1 \leq b$

- The remainder, a_1, is the second digit from the right in the base b expansion of n.

- Continue by successively dividing the quotients by b, obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0.

continued →

Algorithm: Constructing Base *b* Expansions

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$ $\{(a_{k-1} \ldots a_1 a_0)_b$ is base b expansion of $n\}$

- q represents the quotient obtained by successive divisions by b, starting with $q = n$.
- The digits in the base b expansion are the remainders of the division given by q **mod** b.
- The algorithm terminates when $q = 0$ is reached.

Base Conversion

Example: Find the octal expansion of $(12345)_{10}$

Solution: Successively dividing by 8 gives:

- $12345 = 8 \cdot 1543 + 1$
- $1543 = 8 \cdot 192 + 7$
- $192 = 8 \cdot 24 + 0$
- $24 = 8 \cdot 3 + 0$
- $3 = 8 \cdot 0 + 3$

The remainders are the digits from right to left yielding $(30071)_8$.

Comparison of Hexadecimal, Octal, and Binary Representations

TABLE 1 Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Initial 0s are not shown

Each octal digit corresponds to a block of 3 binary digits.
Each hexadecimal digit corresponds to a block of 4 binary digits.
So, conversion between binary, octal, and hexadecimal is easy.

Conversion Between Binary, Octal, and Hexadecimal Expansions

Example: Find the octal and hexadecimal expansions of $(11111010111100)_2$.

Solution:

- To convert to octal, we group the digits into blocks of three $(011\ \ 111\ \ 010\ \ 111\ \ 100)_2$, adding initial 0s as needed. The blocks from left to right correspond to the digits 3,7,2,7, and 4. Hence, the solution is $(37274)_8$.
- To convert to hexadecimal, we group the digits into blocks of four $(0011\ \ 1110\ \ 1011\ \ 1100)_2$, adding initial 0s as needed. The blocks from left to right correspond to the digits 3,E,B, and C. Hence, the solution is $(3EBC)_{16}$.

Binary Addition of Integers

- Algorithms for performing operations with integers using their binary expansions are important as computer chips work with binary numbers. Each digit is called a *bit*.

procedure *add*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}
$c_{prev} := 0$ (represents *carry* from the previous bit addition)
for $j := 0$ to $n - 1$
    $c := \lfloor (a_j + b_j + c_{prev})/2 \rfloor$ - quotient (*carry* for the next digit of the sum)
    $s_j := a_j + b_j + c_{prev} - 2c$ - remainder (j-th digit of the sum)
    $c_{prev} := c$
$s_n := c$
return $(s_n, \ldots, s_1, s_0)$ {the binary expansion of the sum is $(s_n, s_{n-1}, \ldots, s_0)_2$}

$$a_0 + b_0 = c_0 \cdot 2 + s_0$$
$$a_1 + b_1 + c_0 = c_1 \cdot 2 + s_1$$
$$\cdots$$
$$a_j + b_j + c_{j-1} = c_j \cdot 2 + s_j$$

- The number of additions of bits used by the algorithm to add two n-bit integers is $O(n)$.

Binary Multiplication of Integers

- Algorithm for computing the product of two n bit integers.

$$a \cdot b \quad = \quad a \cdot (b_k 2^k + b_{k-1} 2^{k-1} + \ldots + b_1 2 + b_0) \quad = \quad a\, b_k 2^k + a\, b_{k-1} 2^{k-1} + \ldots + a\, b_1 2 + a\, b_0$$

shift by k shift by k-1 shift

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}
for $j := 0$ to $n - 1$
    if $b_j = 1$ **then** $c_j = a$ shifted j places
    else $c_j := 0$
{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ to $n - 1$
  $p := p + c_j$
return p {p is the value of ab}

```
    110     - a
x   101     - b
--------
    1 1 0   - ab0
   000      - ab1
  1 1 0     - ab2
```

- The number of additions of bits used by the algorithm to multiply two n-bit integers is $O(n^2)$.

Primes and Greatest Common Divisors

Section 4.3

Section Summary

- Prime Numbers and their Properties
- Conjectures and Open Problems About Primes
- Greatest Common Divisors and Least Common Multiples
- The Euclidian Algorithm
- gcd(s) as Linear Combinations
- Relative primes

Primes

Definition: A positive integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p.

A positive integer that is greater than 1 and is not prime is called *composite*.

Example: The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

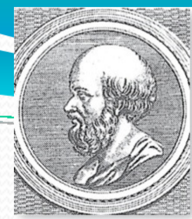
The Fundamental Theorem of Arithmetic (prime factorization)

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

$$\boxed{a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}}$$

Examples:

- $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$
- $641 = 641$
- $999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$
- $1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$



Erastothenes
(276-194 B.C.)

The Sieve of Erastosthenes

- The *Sieve of Erastosthenes* can be used to find all primes not exceeding a specified positive integer.
- For example, consider the list of integers between 1 and 100:

  a. Delete all the integers, other than 2, divisible by 2.
  b. Delete all the integers, other than 3, divisible by 3.
  c. Next, delete all the integers, other than 5, divisible by 5.
  d. Next, delete all the integers, other than 7, divisible by 7.

all remaining numbers between 1 and 100 are prime:

{2,3,7,11,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89, 97}

Why does this work?

continued →

The Sieve of Erastosthenes

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If an integer n is a composite integer, then it must have a prime divisor less than or equal to $\sqrt{n}$.

To see this, note that if $n = ab$, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

For n=100 $\sqrt{n}=10$, thus any composite integer ≤ 100 <u>must have</u> prime factors less than 10, that is 2,3,5,7. The remaining integers ≤ 100 are prime.

Trial division, a <u>very inefficient</u> method of determining if a number n is prime, is to try every integer $i \le \sqrt{n}$ and see if n is divisible by i.

Infinitude of Primes

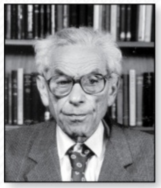
Euclid
(325 B.C.E. – 265 B.C.E.)

Theorem: There are infinitely many primes.

Proof: Assume finitely many primes: $p_1, p_2, \ldots.., p_n$

- Let $q = p_1 p_2 \cdots p_n + 1$
- Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
  - But none of the primes p_j divides q since if $p_j \mid q$, then p_j divides
    $q - p_1 p_2 \cdots p_n = 1$ (contradiction to divisibility by p_j).
  - Hence, there is a prime not on the list $p_1, p_2, \ldots.., p_n$. It is either q, or if q is composite, it is a prime factor of q. This contradicts the assumption that $p_1, p_2, \ldots.., p_n$ are all the primes.
- Consequently, there are infinitely many primes. ◀

This proof was given by Euclid *The Elements*. The proof is considered to be one of the most beautiful in all mathematics. It is the first proof in *The Book,* inspired by the famous mathematician Paul Erdős' imagined collection of perfect proofs maintained by God.

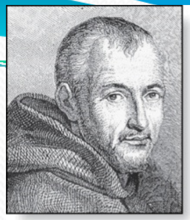


Paul Erdős
(1913-1996)

Generating Primes

- The problem of generating large primes is of both theoretical and practical interest.
- Finding large primes with hundreds of digits is important in cryptography.
- So far, no useful closed formula that always produces primes has been found. There is no simple function $f(n)$ such that $f(n)$ is prime for all positive integers n.
- $f(n) = n^2 - n + 41$ is prime for all integers 1,2,…, 40. Because of this, we might conjecture that $f(n)$ is prime for all positive integers n. But $f(41) = 41^2$ is not prime.
- More generally, there is no polynomial with integer coefficients such that $f(n)$ is prime for all positive integers n.
- Fortunately, we can generate large integers which are almost certainly primes.

Mersenne Primes



Marin Mersenne
(1588-1648)

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called *Mersenne primes*.

- $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, and $2^7 - 1 = 127$ are Mersenne primes.
- $2^{11} - 1 = 2047$ is not a Mersenne prime since $2047 = 23 \cdot 89$.
- There is an efficient test for determining if $2^p - 1$ is prime.
- The largest known prime numbers are Mersenne primes.
- On December 26 2017, 50-th Mersenne primes was found, it is $2^{77,232,917} - 1$, which is the largest Marsenne prime known. It has more than 23 million decimal digits.
- The *Great Internet Mersenne Prime Search* (*GIMPS*) is a distributed computing project to search for new Mersenne Primes.

http://www.mersenne.org/

Conjectures about Primes

- Even though primes have been studied extensively for centuries, many conjectures about them are unresolved, including:
- *Goldbach's Conjecture*: Every even integer n, $n > 2$, is the sum of two primes. It has been verified by computer for all positive even integers up to $1.6 \cdot 10^{18}$. The conjecture is believed to be true by most mathematicians.
- There are infinitely many primes of the form $n^2 + 1$, where n is a positive integer. But it has been shown that there are infinitely many primes of the form $n^2 + 1$ which are the product of at most two primes.
- *The Twin Prime Conjecture*: there are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers $65{,}516{,}468{,}355 \cdot 2^{333{,}333} \pm 1$, which have 100,355 decimal digits.

Greatest Common Divisor (gcd)

Definition: Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and also $d \mid b$ is called the greatest common divisor of a and b. The *greatest common divisor* of a and b is denoted by gcd(a,b).

One can find greatest common divisors of small numbers by inspection.

Example: What is the greatest common divisor of 24 and 36?

Solution: gcd(24,26) = 12

Example:What is the greatest common divisor of 17 and 22?

Solution: gcd(17,22) = 1

Greatest Common Divisor (gcd)

Definition: The integers a and b are *relatively prime* if their greatest common divisor is gcd(a,b) = 1.

Example: 17 and 22

Definition: The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Solution: Because $\gcd(10,17) = 1$, $\gcd(10,21) = 1$, and $\gcd(17,21) = 1$, 10, 17, and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: No, since $\gcd(10,24) = 2$.

Finding the Greatest Common Divisor Using Prime Factorizations

- Suppose that (unique) prime factorizations of *a* and *b* are:

$$a = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}, \qquad b = p_1^{b_1} p_2^{b_2} \dots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\boxed{\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \dots p_n^{\min(a_n,b_n)}.}$$

- This formula is valid since the integer on the right (of the equals sign) divides both *a* and *b*. No larger integer can divide both *a* and *b*.

Example: $120 = 2^3 \cdot 3 \cdot 5 \qquad 500 = 2^2 \cdot 5^3$

$\gcd(120,500) = 2^{\min(3,2)} \cdot 3^{\min(1,0)} \cdot 5^{\min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$

- NOTE: finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Least Common Multiple (lcm)

Definition: The least common multiple of the positive integers *a* and *b* is the smallest positive integer that is divisible by both *a* and *b*. It is denoted by lcm(*a*,*b*).

- The least common multiple can also be computed from the prime factorizations.

$$\boxed{\operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}}$$

This number is divided by both *a* and *b* and no smaller number is divided by *a* and *b*.

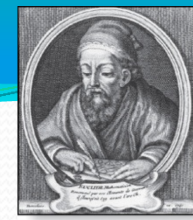
Example: $\operatorname{lcm}(2^3 3^5 7^2,\ 2^4 3^3) = 2^{\max(3,4)}\ 3^{\max(5,3)}\ 7^{\max(2,0)} = 2^4\ 3^5\ 7^2$

- The greatest common divisor (gcd) and the least common multiple (lcm) of two integers are related by:

Theorem 5: Let a and b be positive integers. Then

$$\boxed{a \cdot b = \gcd(a,b) \cdot \operatorname{lcm}(a,b)}$$

Euclidean Algorithm



Euclid
(325 B.C.E. – 265 B.C.E.)

- The Euclidian algorithm is an <u>efficient method</u> for computing the greatest common divisor of two integers. It is based on the idea that gcd(a,b) = gcd(b,r) when $a > b$ and r is the remainder when a is divided by b.

(indeed, since $a = bq + r$, *then* $r = a - b\,q$. *Thus,* if $d|a$ and $d|b$ then $d|r$)

Example: Find gcd(287,91):

- $287 = 91 \cdot 3 + 14$ Divide 287 by 91
- $91 = 14 \cdot 6 + 7$ Divide 91 by 14
- $14 = 7 \cdot 2 + 0$ Divide 14 by 7

Stopping condition

gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = gcd(7, 0) = 7

continued →

Euclidean Algorithm

- The Euclidean algorithm expressed in pseudocode is:

procedure *gcd*(*a, b*: positive integers, WLOG assume a>b)
$x := a$
$y := b$
while $y \neq 0$
  $r := x$ **mod** y
  $x := y$
  $y := r$
return x {gcd(a,b) is x}

- Note: the time complexity of the algorithm is $O(\log b)$, where a > b.

Correctness of Euclidean Algorithm

Lemma 1: Let $r = a \textbf{ mod } b$, where $a \geq b > r$ are integers. Then $\gcd(a,b) = \gcd(b,r)$.

Proof:

- Any divisor or a and b must also be a divisor of r since $a = b\,q + r$ (for quotient $q = a \textbf{ div } b$) and $r = a - b\,q$.
- Therefore, $\gcd(a,b) = \gcd(b,r)$. ◀

Correctness of Euclidean Algorithm

- Suppose that a and b are positive integers with $a \geq b$.
  Let $r_0 = a$ and $r_1 = b$.
  Successive applications of the division algorithm yields:

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \leq r_2 < r_1 \leq r_0, \\
r_1 &= r_2 q_2 + r_3 && 0 \leq r_3 < r_2, \\
&\;\;\vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + \underset{\text{gcd}}{r_n} && 0 \leq r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n .
\end{aligned}
$$

- Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 > r_1 > r_2 > \cdots \geq 0$. The sequence can't contain more than a terms.
- By Lemma 1
  $\gcd(a,b) = \gcd(r_0,r_1) = \cdots = \gcd(r_{n-1},r_n) = \gcd(r_n, 0) = r_n$.
- Hence the gcd is the last nonzero remainder in the sequence of divisions. ◀

Étienne Bézout
(1730-1783)

gcd(s) as Linear Combinations

Bézout's Theorem: If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa + tb$.

Definition: If a and b are positive integers, then integers s and t such that $\gcd(a,b) = sa + tb$ are called *Bézout coefficients* of a and b. The equation $\gcd(a,b) = sa + tb$ is called *Bézout's identity*.

Expression $sa + tb$ is a *linear combination* of a and b with coefficients of s and t.

Example: $\gcd(6,14) = 2 = (-2)\cdot 6 + 1\cdot 14$

Finding gcd(s) as Linear Combinations

Example: Express gcd(252,198) = 18 as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show gcd(252,198) = 18

i. $252 = 1 \cdot 198 + 54$

ii. $198 = 3 \cdot 54 + 36$

iii. $54 = 1 \cdot 36 + 18$

iv. $36 = 2 \cdot 18$

- Working backwards, from iii and i above

  $18 = 54 - 1 \cdot 36$

  $36 = 198 - 3 \cdot 54$

- Substituting the 2nd equation into the 1st yields:

  $18 = 54 - 1 \cdot (198 - 3 \cdot 54) = 4 \cdot 54 - 1 \cdot 198$

- Substituting $54 = 252 - 1 \cdot 198$ (from i)) yields:

  $18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198$

This method illustrated above is a two pass method. It first uses the Euclidian algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers.
A one pass method, called the *extended Euclidean algorithm*, is developed in the exercises.

Consequences of Bézout's Theorem

Lemma 2: If a, b, c are positive integers such that a and b are relatively prime ($\gcd(a, b) = 1$) and $a \mid bc$ then $a \mid c$.

Proof: Assume $\gcd(a, b) = 1$ and $a \mid bc$

- Since $\gcd(a, b) = 1$, by Bézout's Theorem there are integers s and t such that
$$sa + tb = 1.$$
- Multiplying both sides of the equation by c, yields $sac + tbc = c$.
- From Theorem 1 of Section 4.1:
$a \mid bc$ implies $a \mid tbc$ (part ii). Since $a \mid sac$ then a divides $sac + tbc$ (part i).
We conclude $a \mid c$, since $sac + tbc = c$. ◀

A generalization of Lemma 2 below is important for proving uniqueness of *prime factorization*:

Lemma 3: If p is prime and $p \mid a_1 a_2 \dots a_n$ where a_i are integers then $p \mid a_i$ for some i.

Dividing Congruences by an Integer

- Dividing both sides of a valid congruence by an integer does not always produce a valid congruence (see Section 4.1).

- But dividing by an integer relatively prime to the modulus does produce a valid congruence:

Theorem 7: Let m be a positive integer and let *a*, *b*, and *c* be integers. If $\gcd(c,m) = 1$ and $ac \equiv bc \pmod{m}$, then

$$a \equiv b \pmod{m}.$$

NOTE: can always divide congruency by any prime number $p > \sqrt{m}$ since $\gcd(p,m) = 1$

Proof: Since $ac \equiv bc \pmod{m}$, $m \mid ac - bc = c(a - b)$ by Lemma 2 and the fact that $\gcd(c,m) = 1$, it follows that $m \mid a - b$. Hence, $a \equiv b \pmod{m}$. ◀