ECCV 2006 tutorial on

Graph Cuts vs. Level Sets

part III

Connecting Graph Cuts and Level Sets

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Graph Cuts versus Level Sets

- Part I: Basics of *graph cuts*
- Part II: Basics of *level-sets*
- Part III: Connecting *graph cuts* and *level-sets*
- Part IV: Global vs. local optimization algorithms

Graph Cuts versus Level Sets

- Part III: Connecting graph cuts and level sets

  - Minimal surfaces, global and local optima
  - Integral and differential approaches
  - Learning and shape prior in graph cuts and level-sets

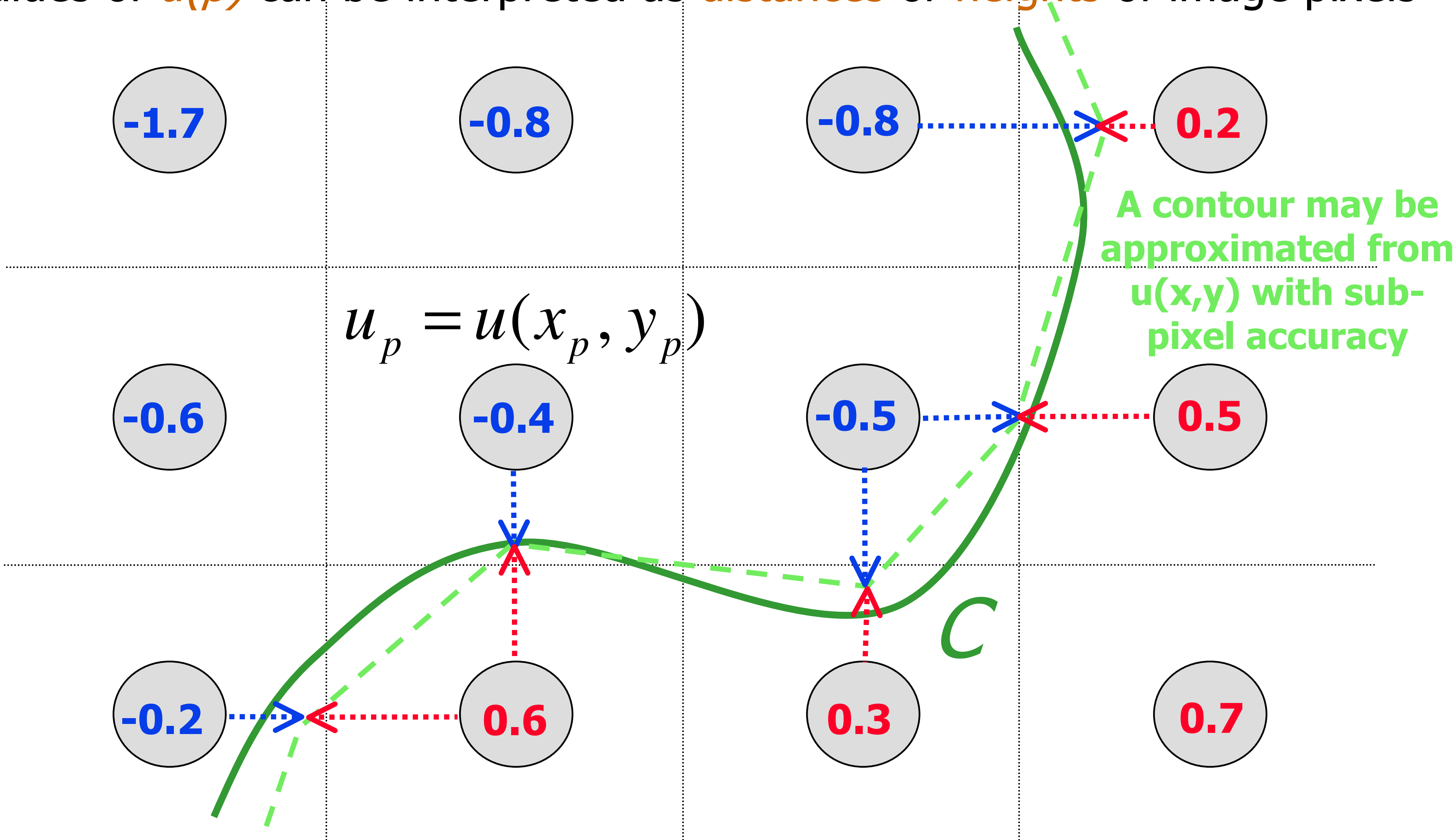
Connecting graph cuts and level sets

- Integral and differential approaches
  - Integral vs. differential geometry
    - Implicit surface representation via level sets and graph cuts
    - Sub-pixel accuracy vs. non-deterministic surface
  - Differential and integral solutions for surface evolution PDEs
    - Gradient flow as a sequence of optimal small step
    - L2 distance between contours/surfaces
    - PDE-cuts (pluses and minuses)
    - Spatio-temporal approach
    - Shortcomings of narrow band cuts and DP snakes

Integral and differential approaches:

Implicit (region-based) surface representation

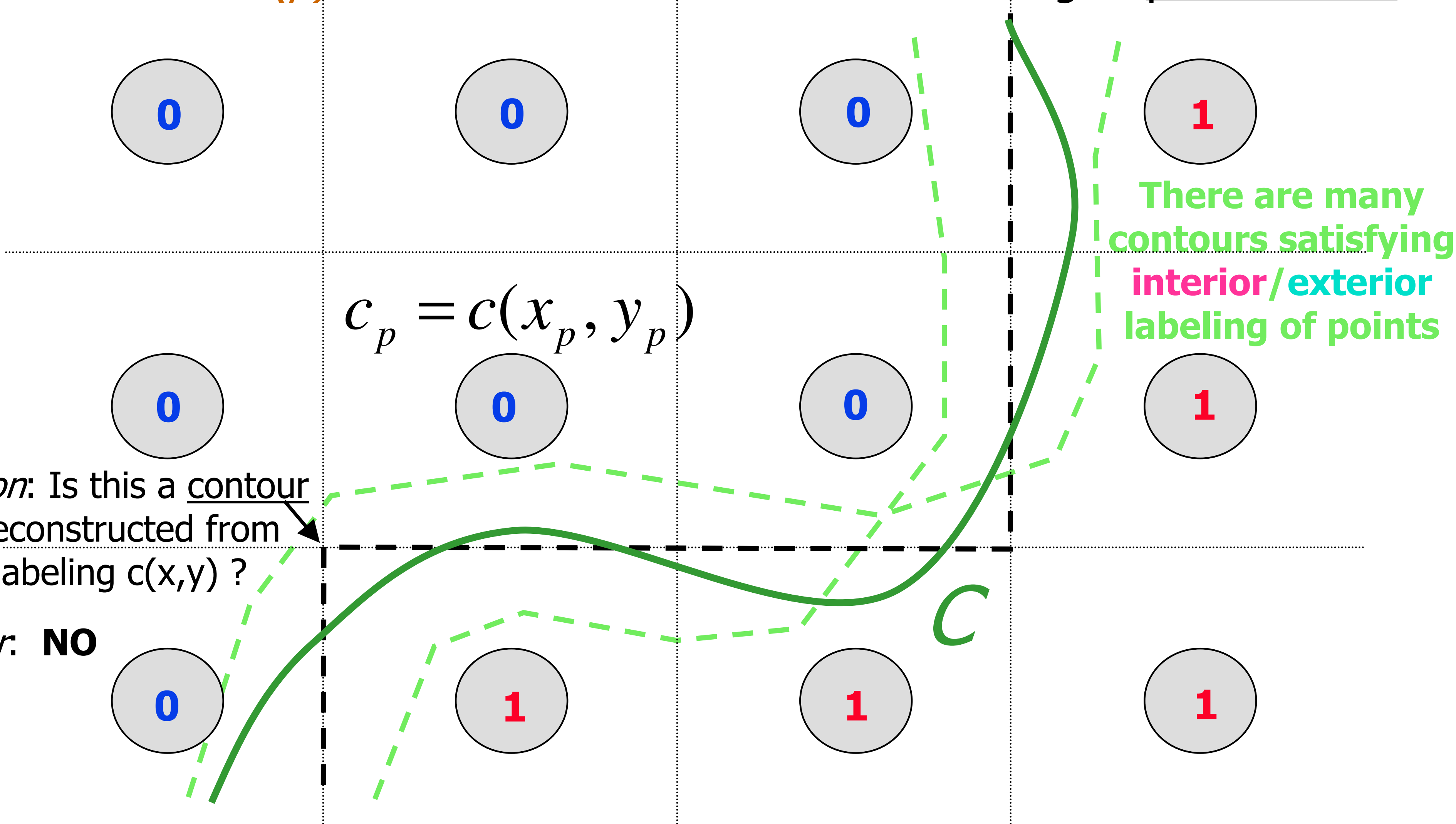
- Level set function *u(p)* is normally stored on image pixels
- Values of *u(p)* can be interpreted as distances or heights of image pixels



Integral and differential approaches:

Implicit (region-based) surface representation

- Graph cuts represent surfaces via binary function *c(p)* on image pixels
- Two values of *c(p)* indicate interior and exterior labeling of pixel centers



$$c_p = c(x_p, y_p)$$

Question: Is this a contour to be reconstructed from binary labeling c(x,y) ?

Answer: **NO**

Integral and differential approaches:

Implicit (region-based) surface representation

- Both level-sets and graph cuts use region-based implicit representation of contours

- Level-set function *u(p)* allows to approximately reconstruct a contour with *sub-pixel accuracy*
- Graph cuts use a “non-deterministic” representation of contours. No particular contour satisfying given pixel labeling is fixed

Integral and differential approaches:

Sub-pixel accuracy

- Level-set function *u(p)* allows to approximately restore a contour
  - with "*sub-pixel accuracy*"
- Graph cuts do not identify any particular contour among those that satisfy the pixel labeling
  - no "*sub-pixel accuracy*"

Integral and differential approaches:

Sub-pixel accuracy,… what for?

- "Super Resolution"
  - … if original data does not have sufficient resolution.
- In any case, one can use a regular grid of acceptable resolution which can be either finer or courser than the data.
- Now-days images often have fairly high resolution and pixel-size segmentation accuracy is more than enough for many applications.

Integral and differential approaches:

Sub-pixel accuracy,… who cares, who does not, and why?

- Level-sets need sub-pixel accuracy for a technical reason:
  - Explicit estimation of contour derivatives (e.g. curvature) is an intrinsic part of variational optimization techniques of differential geometry

e.g. curvature flow equation

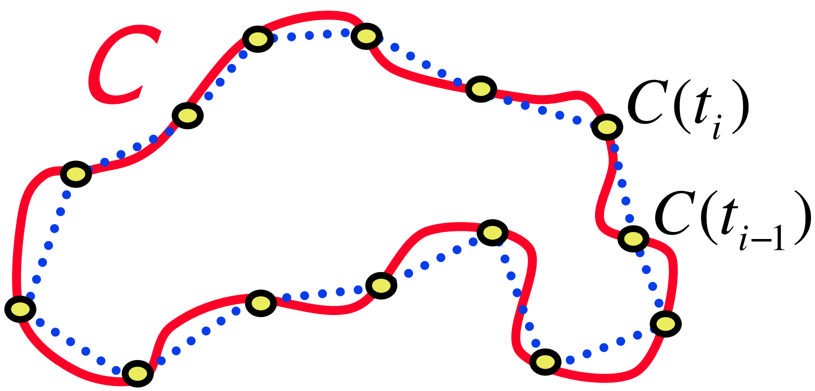
$$C_t = \kappa \cdot \vec{N} \quad \Rightarrow \quad \frac{\partial u}{\partial t} = \kappa \cdot |\nabla u|$$

explicit (*snakes*) — implicit (*level-sets*)

- Graph cuts methods DO NOT use any surface derivatives in their inner workings
  - **sub-pixel accuracy is unnecessary for graph cuts to work**

Integral and differential approaches:

Contour length in differential geometry?



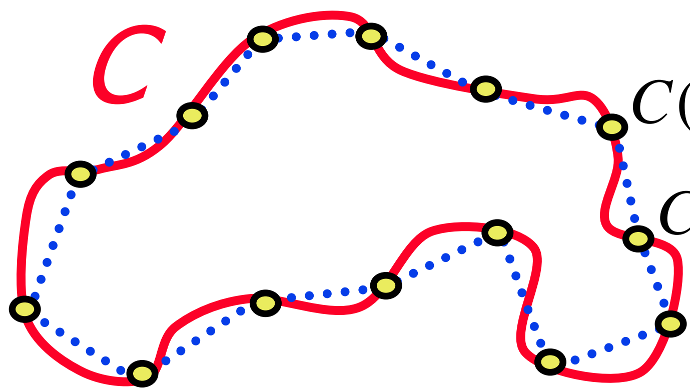
$$C(t) \; : \; [0,1] \rightarrow R^2$$

$$\| C \|_\varepsilon = \sup \left\{ \sum_{i=1}^{n} \| C(t_i) - C(t_{i-1}) \|_\varepsilon \; : \; n > 0, \; 0 \le t_0 \le t_1 \le \ldots \le t_n \le 1 \right\}$$

- Limit of *finite differences* approximation

Integral and differential approaches:

Contour length in differential geometry?



$$C(t) \; : \; [0,1] \rightarrow R^2$$

If $\quad C'_{t_0} = \lim_{t \to t_0} \frac{\| C(t) - C(t_0) \|_\varepsilon}{| t - t_0 |}$ then $\quad \| C \|_\varepsilon = \int_0^1 C'_t \cdot dt$

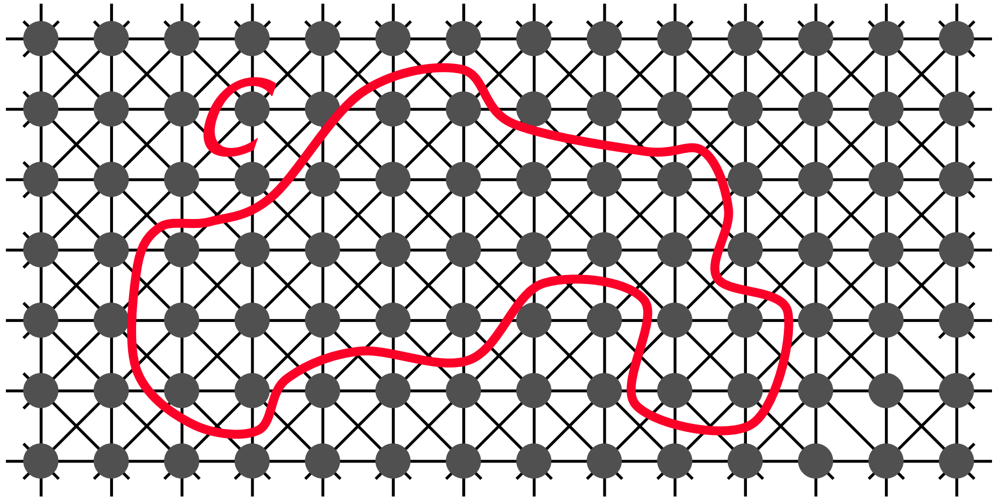
- This is standard ***Differential Geometry*** approach to length
- Variational optimization gives standard *mean curvature flow*

$$\frac{dC}{dt} = \kappa \cdot \vec{N} \quad \Rightarrow \quad \frac{du}{dt} = \kappa \cdot | \nabla u |$$

as in level-sets

Integral and differential approaches:

How do *graph cuts* evaluate contour length?

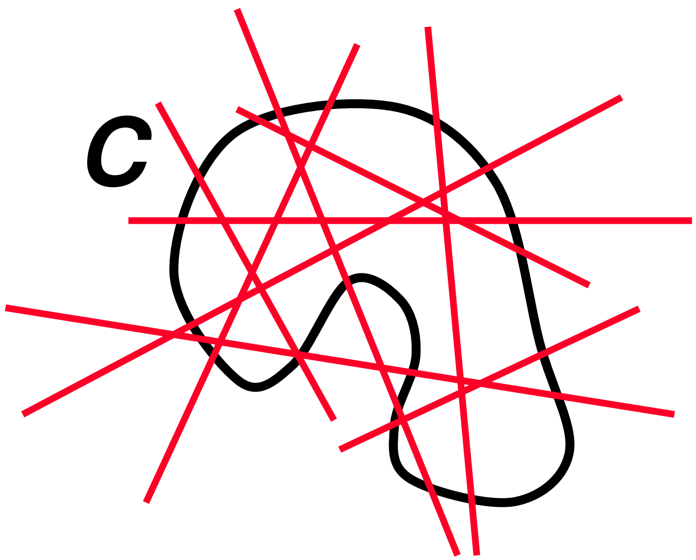


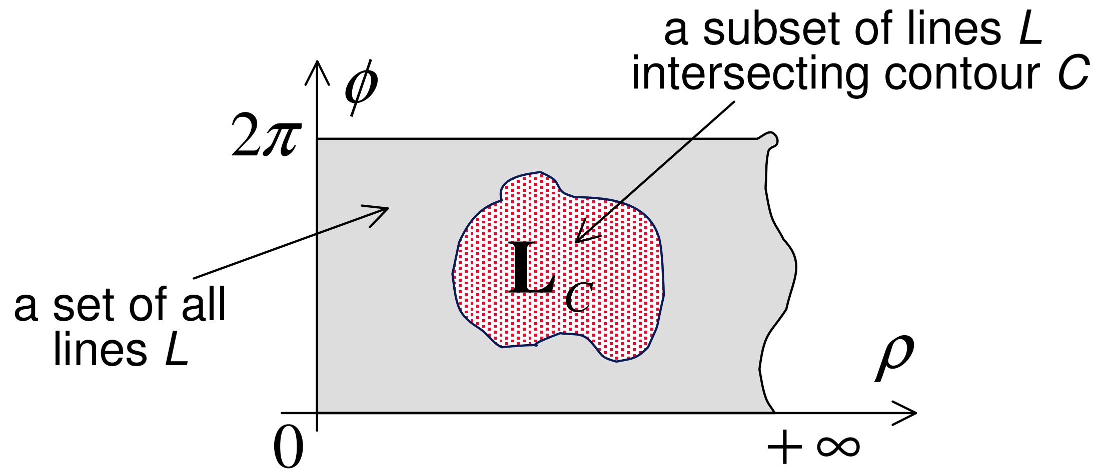
$$\| C \| = \sum_{e \in C} | e |$$

- As mentioned earlier, the cost of a cut can approximate geometric length of contour *C* [Boykov&Kolmogorov, ICCV 2003]
- This result fundamentally relies on ideas of ***Integral Geometry*** (also known as ***Probabilistic Geometry***) originally developed in 1930's.
  - e.g. **Blaschke**, **Santalo**, **Gelfand**

Integral and differential approaches:

Integral geometry approach to *length*





probability that a "randomly drown" line intersects *C*

Euclidean length of ***C*** :

$$\| C \|_{\varepsilon} = \frac{1}{2} \int n_L \cdot d\rho \cdot d\phi$$

Cauchy-Crofton formula

the number of times line *L* intersects *C*

Integral and differential approaches:

Graph cuts and integral geometry

Graph nodes are imbedded in R2 in a grid-like fashion

Edges of any regular neighborhood system generate families of lines

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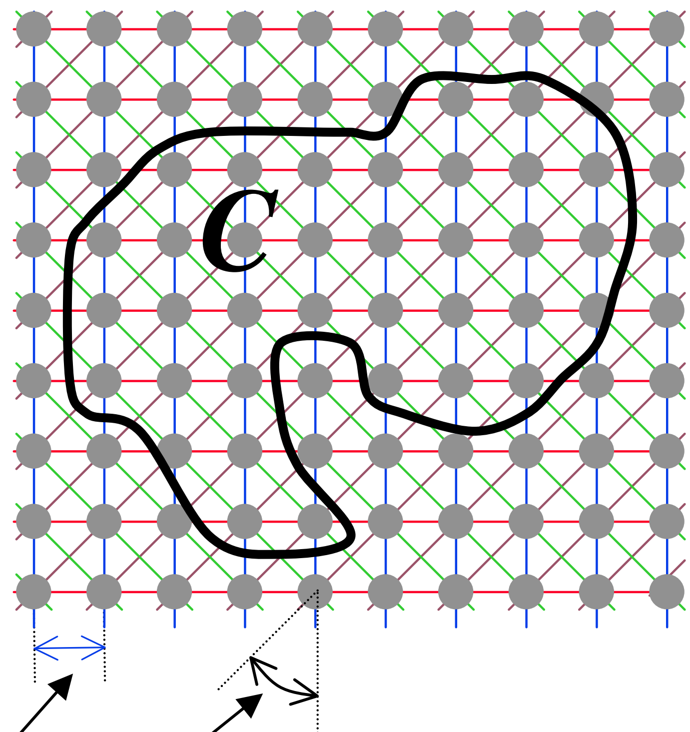


$$\|C\|_{\varepsilon} \approx \frac{1}{2}\sum_k n_k \cdot \Delta\rho_k \cdot \Delta\phi_k = \|C\|_{gc}$$

Euclidean length

the number of edges of family k intersecting *C*

graph cut cost for edge weights:

$$w_k = \frac{\Delta\rho_k \cdot \Delta\phi_k}{2}$$

Length can be estimated without computing any derivatives

Differential vs. integral approach to length

Differential geometry

$$\| C \|_{\varepsilon} = \int_0^1 C_t' \cdot dt$$

Parametric contour representation

$$\| C \|_{\varepsilon} = \int_{\Omega} | \nabla u | \, dx$$

Level-set function representation

Integral geometry

$$\| C \|_{\varepsilon} = \tfrac{1}{2} \int n_L \cdot d\rho \cdot d\phi$$

Cauchy-Crofton formula

Integral and differential approaches:

Graph cuts and integral geometry

- Min-cut/max-flow algorithms find **globally optimal cut**
- In the most general case of directed graphs, a cost of n-links is a linear combination of geometric length and flux of a given vector field

  e.g. *Riemannian*

  while t-links can implement any regional bias

[Boykov&Kolmogorov, ICCV 2003]

[Kolmogorov&Boykov, ICCV 2005]

Integral and differential approaches:

From global to local optimization

- In some problems local minima is desirable
  - when global minima is a trivial solution
  - when a good initial solution is known
  - many "shape prior" techniques rely on intermediate solutions (Daniel will explain more)

differential approach

- **Level-sets** is a variational optimization technique computing *gradient flow* evolution of contours converging to a local minima.

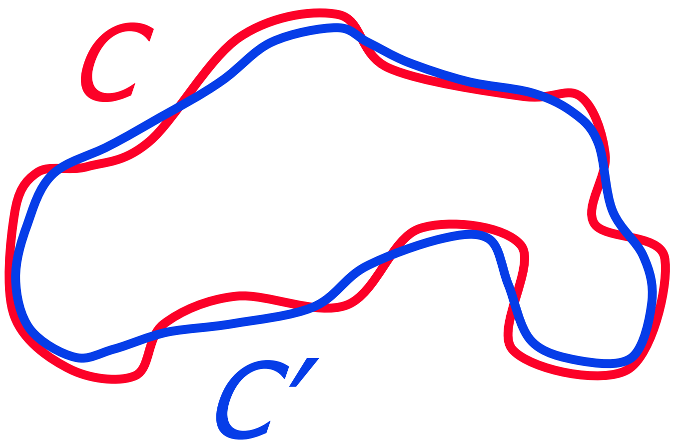
integral approach

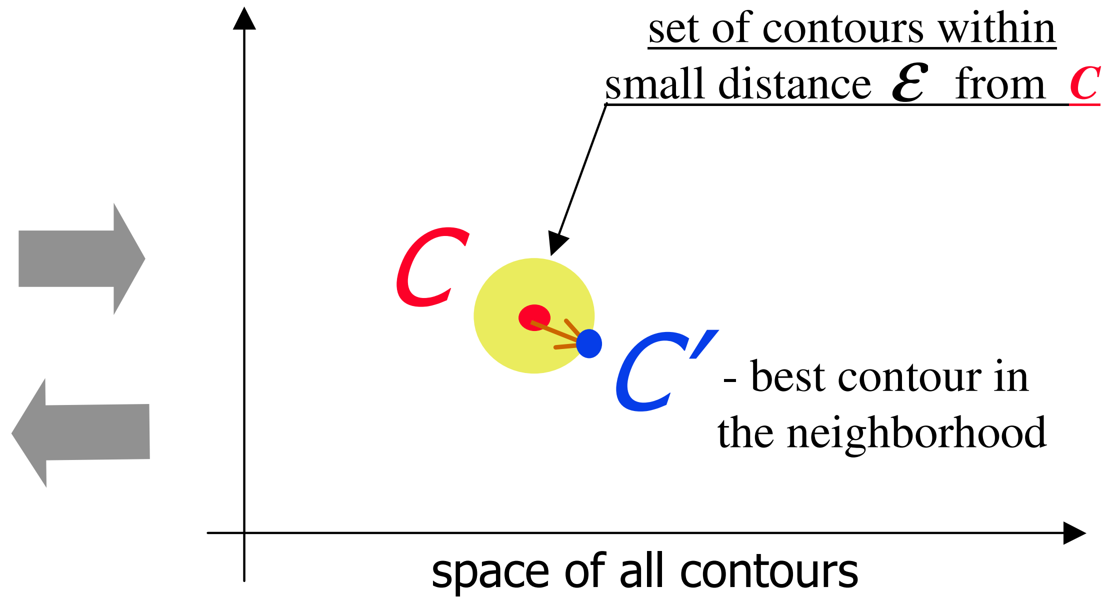
- In fact, **graph cuts** can be also converted into a local optimization method.

Integral and differential approaches:

Gradient flow of a contour for energy *F(C)*

- Contour *C* is a point in the space of all contours



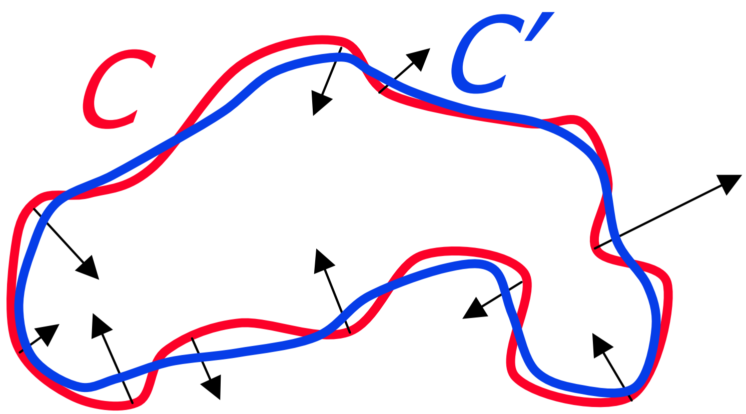


- *Gradient flow* evolution implies infinitesimal step in the space of contours giving the largest energy decrease among all small steps of the same size

Integral and differential approaches:

Differential approach to *gradient flow*

- Level-sets and other *differential methods* for computing *gradient flow* of a contour explicitly estimate local motion (speed) at each point



Local speed could be proportional to local *curvature*

e.g. *mean curvature flow* minimizing Euclidean length

$$\frac{dC}{dt} = \kappa \cdot \vec{N}$$

explicit (*snakes*)

and

$$\frac{\partial u}{\partial t} = \kappa \cdot |\nabla u|$$

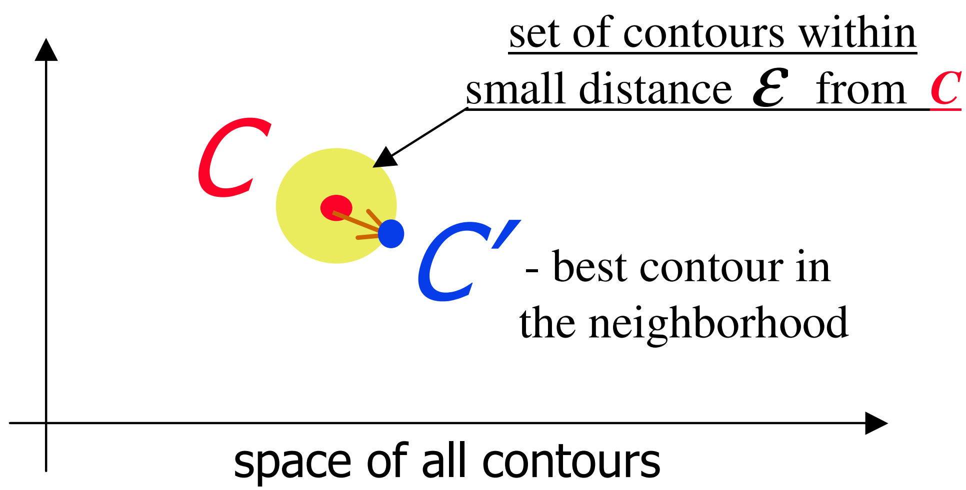
implicit (*level-sets*)

Integral and differential approaches:

Integral approach to *gradient flow*

- Discrete and continuous max-flow algorithms can "directly" compute an optimal step *C'* in the small neighborhood of *C*.

- *integral* approach to estimating contour evolution.



Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

Measuring distance between contours

- What is a small "neighborhood" of contour C ?

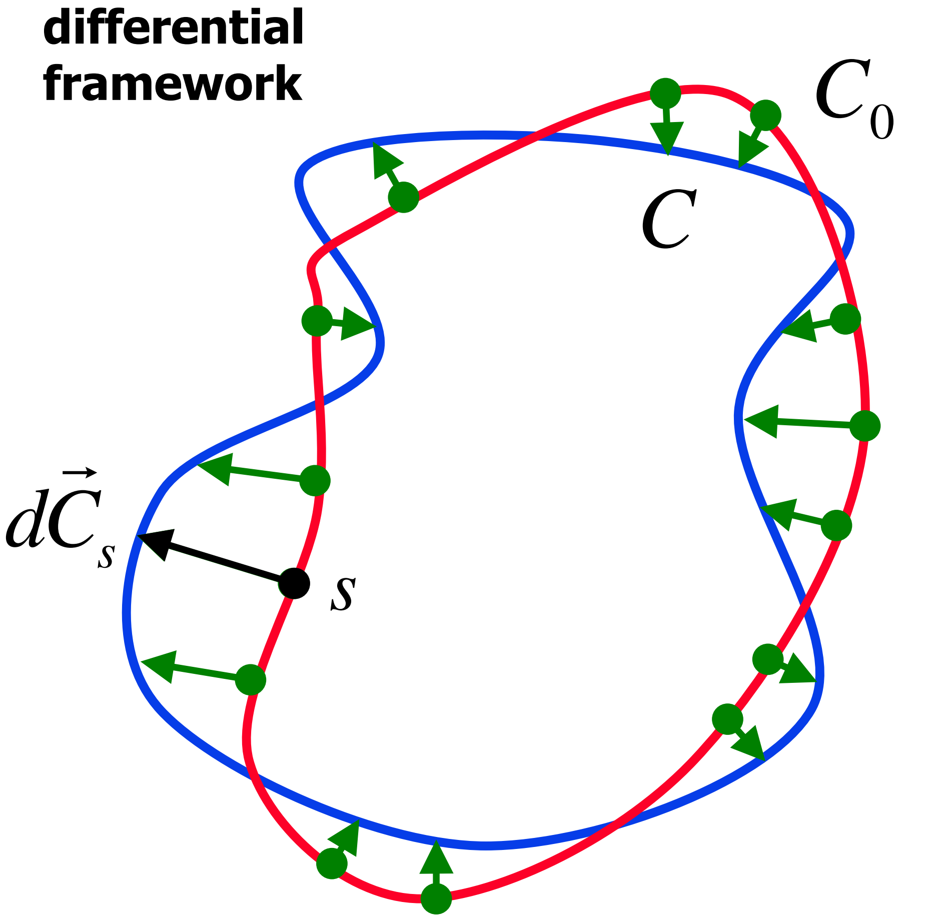
$$\| C - C' \| \leq \varepsilon$$

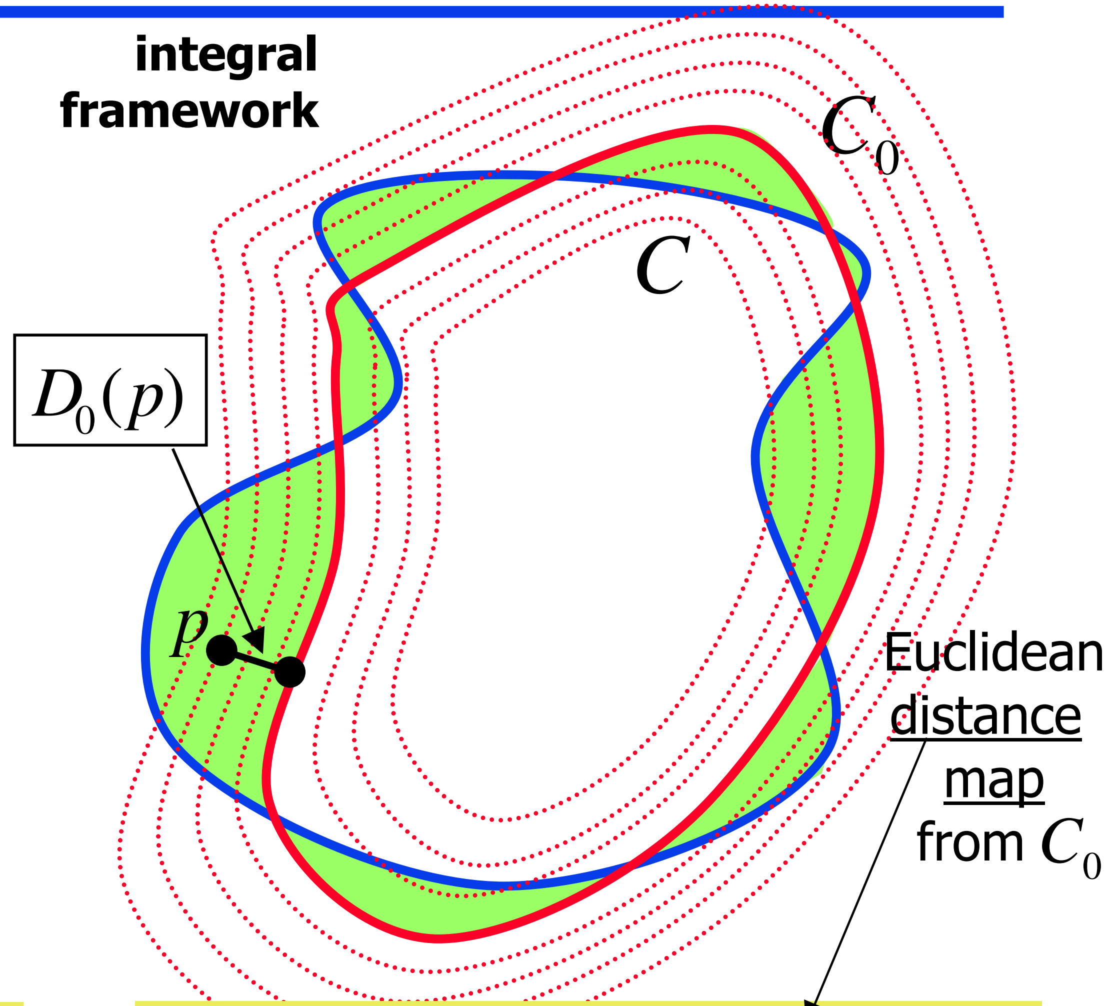
- Typically, gradient flow is based on L_2 metric in the space of contours

Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

Measuring L_2 distance between contours





$$dist^2(C,C_0) = \langle dC, dC \rangle = \int_{C_0} |dC_s|^2 ds$$

$$dist^2(C,C_0) = 2 \cdot \int_{\Delta C} D_0(p)\, dp$$

Boykov, Kolmogorov, Cremers, Delong, ECCV ’06

Integral and differential approaches:

Integral approach to gradient flow

$$F(C)$$ surface functional (energy)

$$\min_{C:dist(C,C_0)=\varepsilon} F(C)$$ gradient flow step from C_0

$$\min_{C} F(C) + \underline{\lambda \cdot dist^2(C,C_0)}$$ unconstrained optimization with *Lagrangian multiplier*

- Penalty for moving away from the current position
  - converts global optimization of *F(C)* into *gradient descent (flow)*
- There is a connection between λ and *time*

Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

Integral approach to gradient flow

$$E(C) \quad = \quad F(C) + \frac{1}{2(t-t_0)} \cdot dist^2(C, C_0)$$

Minimization of this energy is equivalent to solving a standard gradient flow equation

$$0 = \frac{dE}{dC} = \frac{dF}{dC} + \frac{(C-C_0)}{(t-t_0)}$$

$$t \rightarrow t_0 \qquad \Rightarrow \qquad \frac{dC}{dt} = -\frac{dF}{dC}$$

E(C) can be minimized globally via discrete or continuous max-flow algorithms

Boykov, Kolmogorov, Cremers, Delong, ECCV ’06

Integral and differential approaches:

PDE cuts

Compute minimum cut for different values of time parameter *t*

$$E(C) \quad = \quad F(C) + \frac{1}{2(t-t_0)} \cdot dist^2(C, C_0)$$

- A sequence of cuts $C_0, C_1, C_2, \ldots, C_n$
- Transition times $t_0, t_1, t_2, \ldots, t_n$

$$F(C_0) > F(C_1) > F(C_2) > \ldots > F(C_n)$$

Initial solution

smallest detectable step

global minima

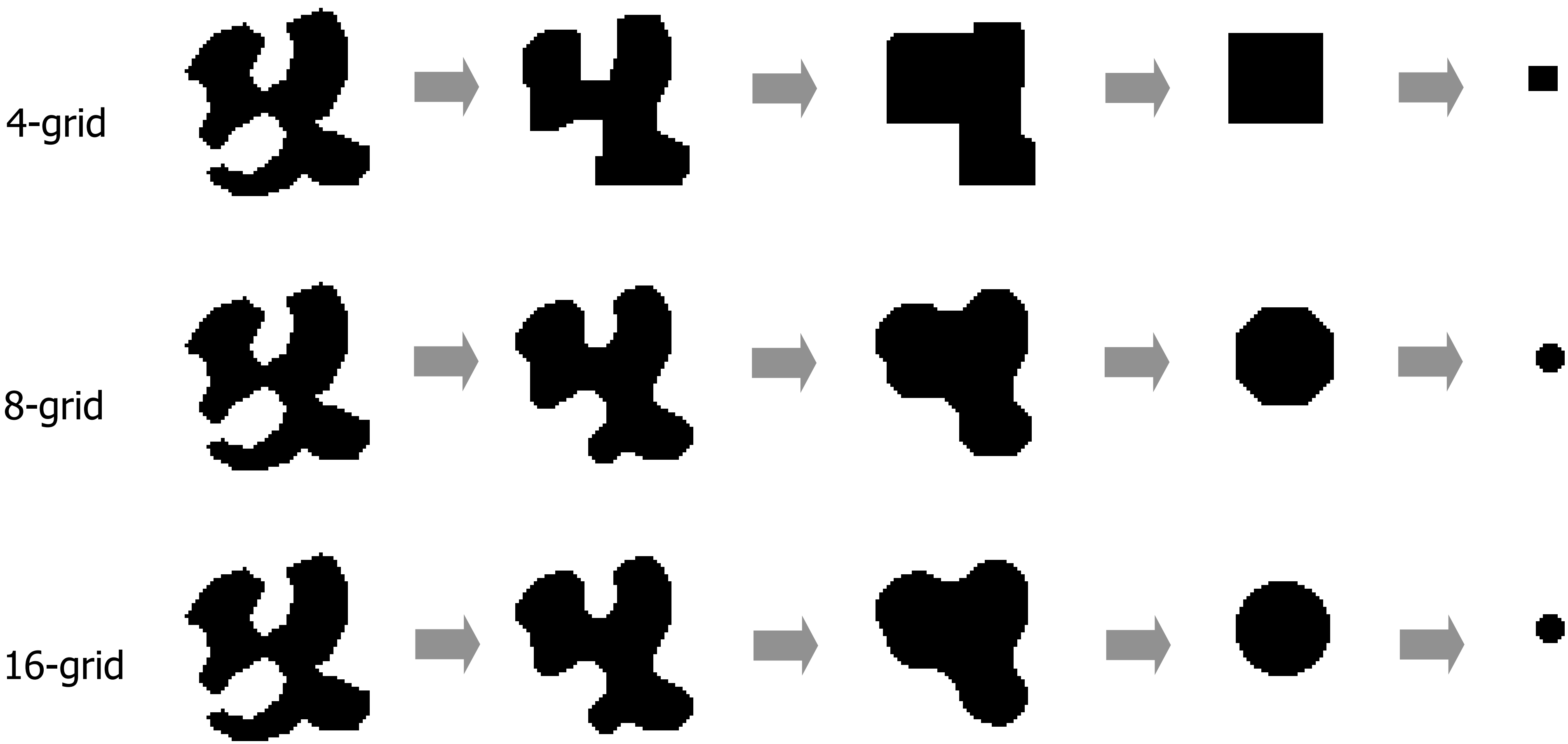
Local minima criteria: ……………………

Integral and differential approaches:

Gradient flows via discrete graph cuts

$$F(C) = \| C \|_\varepsilon$$

Under mean curvature motion any contour should converge to a circle before collapsing into a point



Boykov, Kolmogorov, Cremers, Delong, ECCV '06

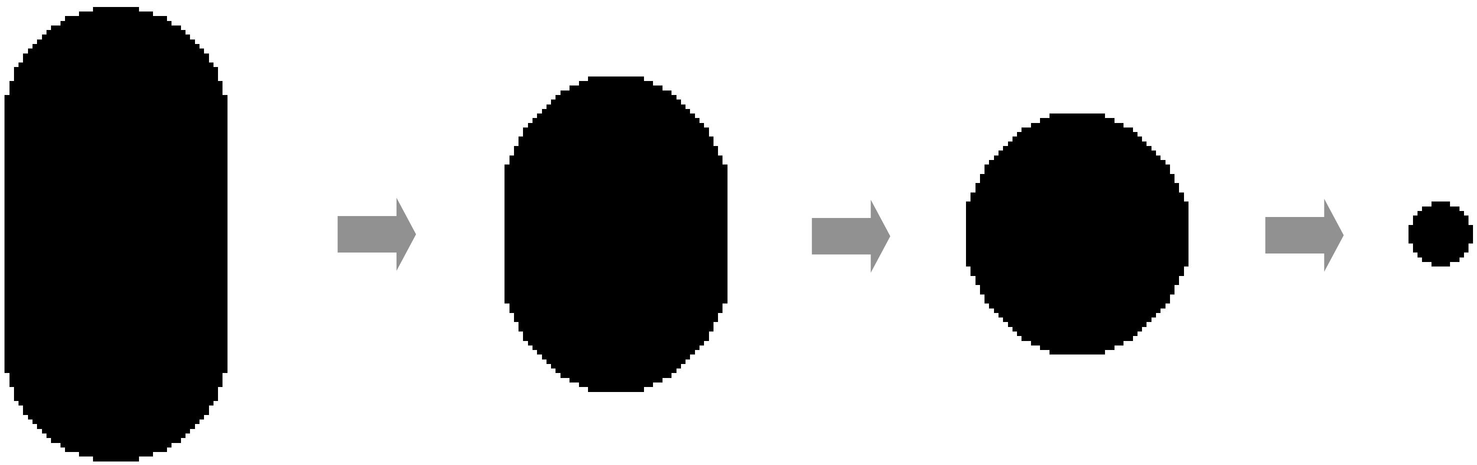
Integral and differential approaches:

Gradient flows via discrete graph cuts

$$F(C) = \| C \|_{\varepsilon}$$

Under mean curvature motion a point on a contour Moves with a speed proportional to local curvature

16-grid

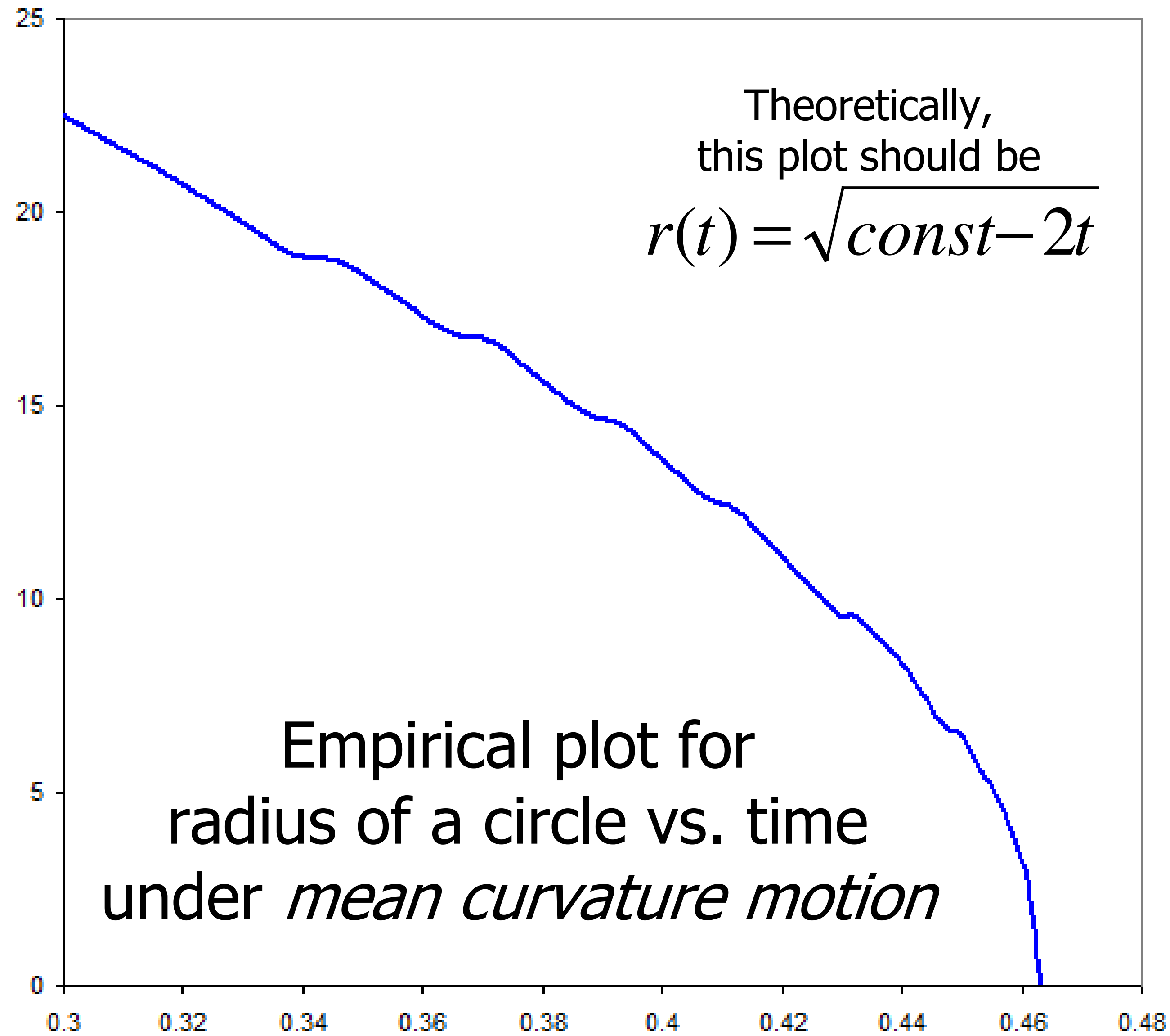


NOTE: straight sides of the sausage should not move until the sausage collapses into a circle from the top and the bottom

Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

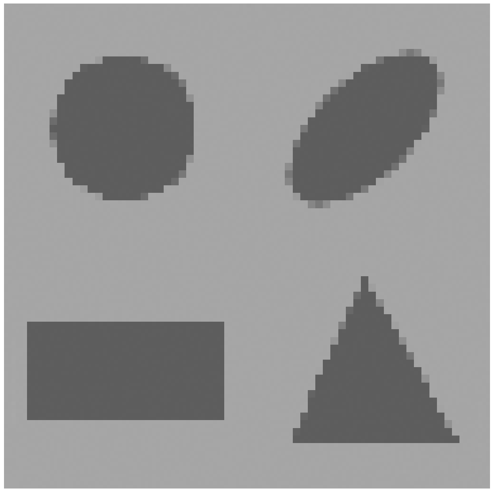
Gradient flows via discrete graph cuts



Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

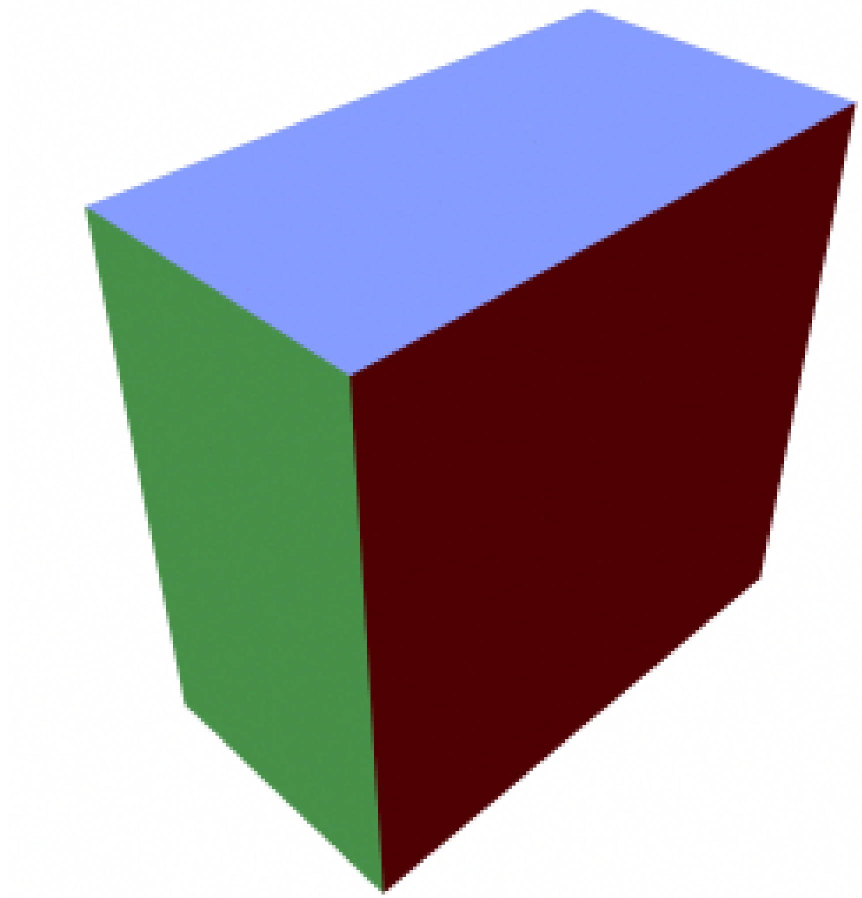
PDE cuts for image based metric



Integral and differential approaches:

Gradient flows via discrete graph cuts

$F(C) = \| C \|_{\varepsilon}$ **mean curvature motion in 3D**



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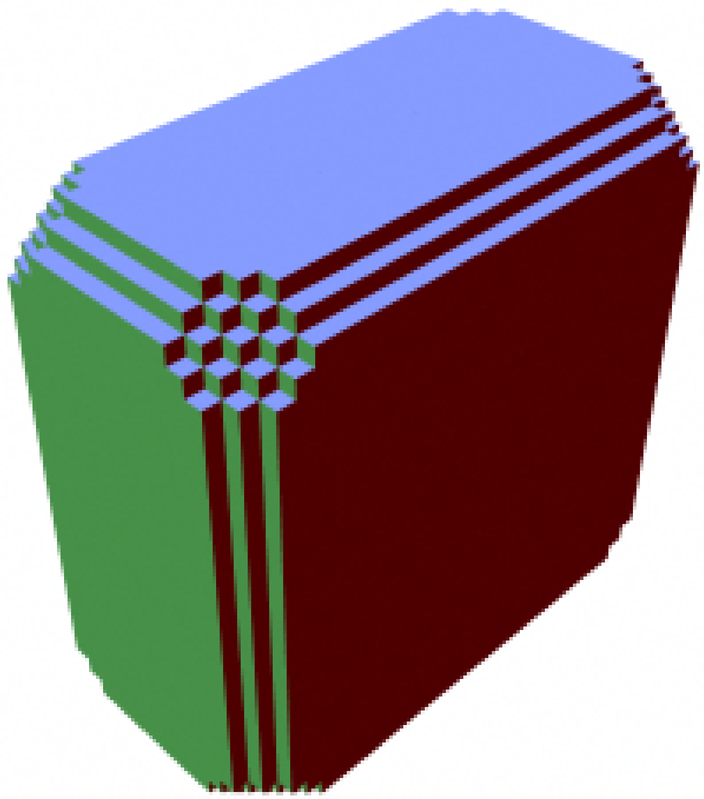

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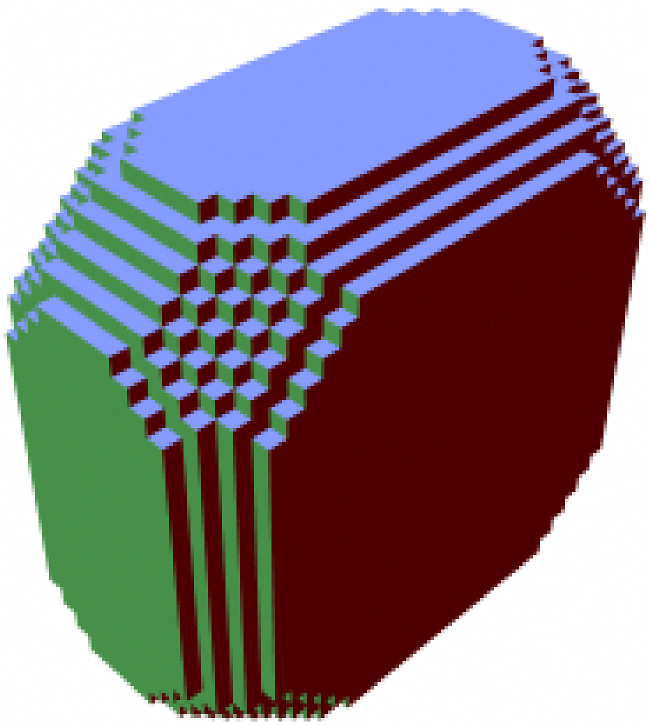

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Gradient flows via discrete graph cuts

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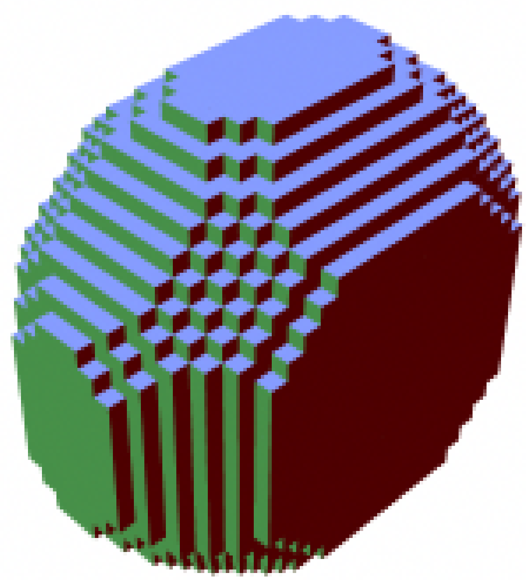
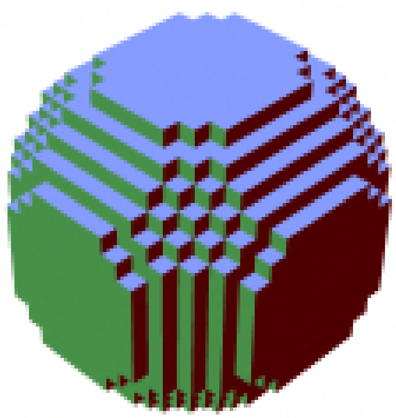


Integral and differential approaches:

Gradient flows via discrete graph cuts

$$F(C) = \| C \|_{\varepsilon}$$

mean curvature motion in 3D



Integral and differential approaches:

Gradient flows via discrete graph cuts

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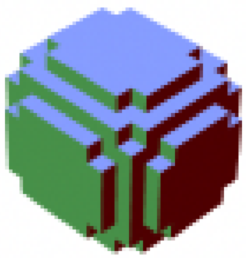

Integral and differential approaches:

Gradient flows via discrete graph cuts

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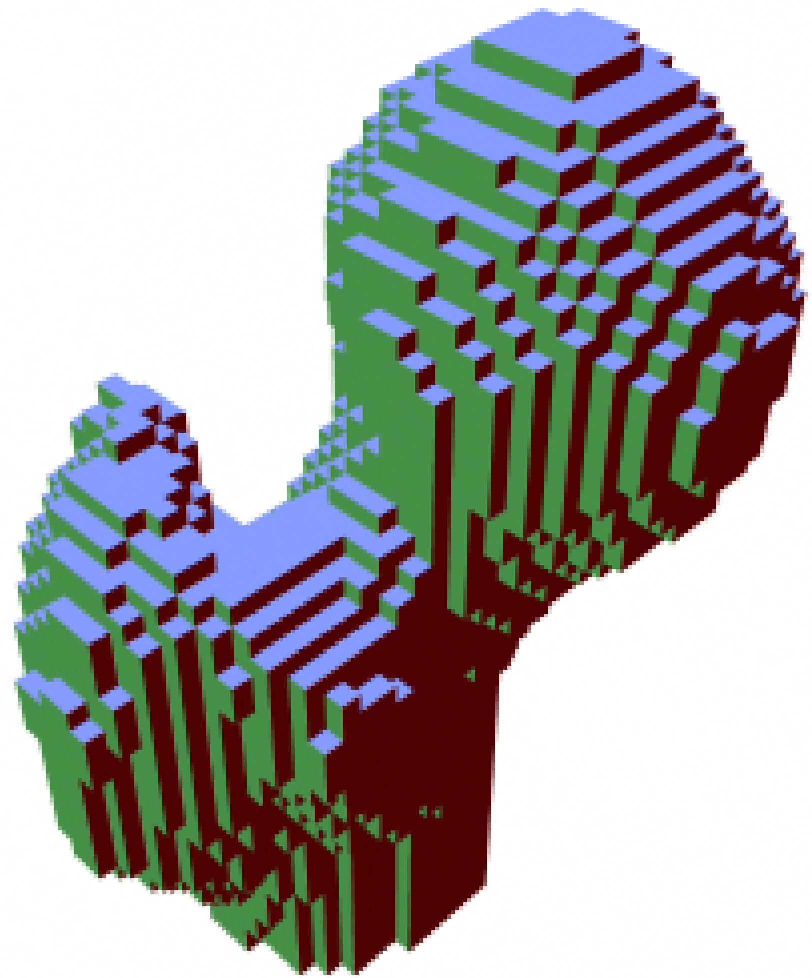
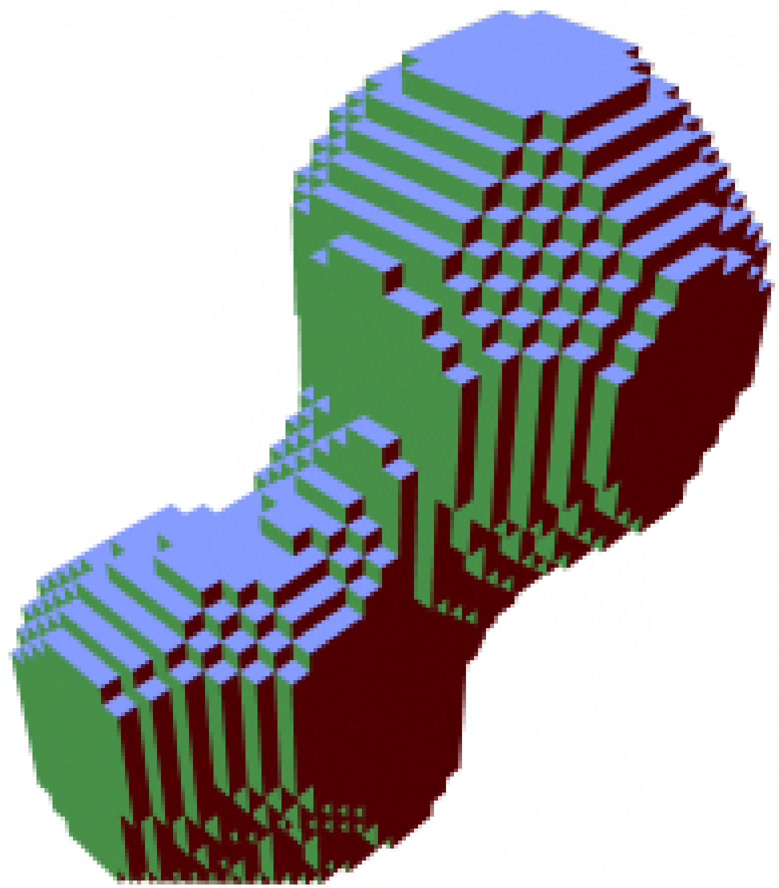


Integral and differential approaches:

Gradient flows via discrete graph cuts

$$F(C) = \| C \|_{\varepsilon}$$

mean curvature motion in 3D



Integral and differential approaches:

Gradient flows via discrete graph cuts

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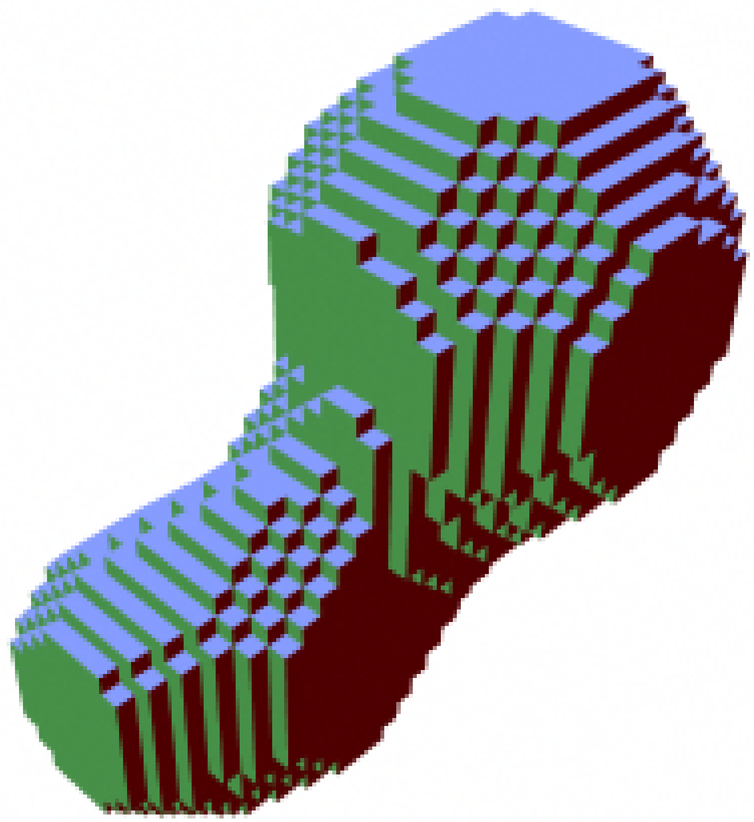

Integral and differential approaches:

Gradient flows via discrete graph cuts

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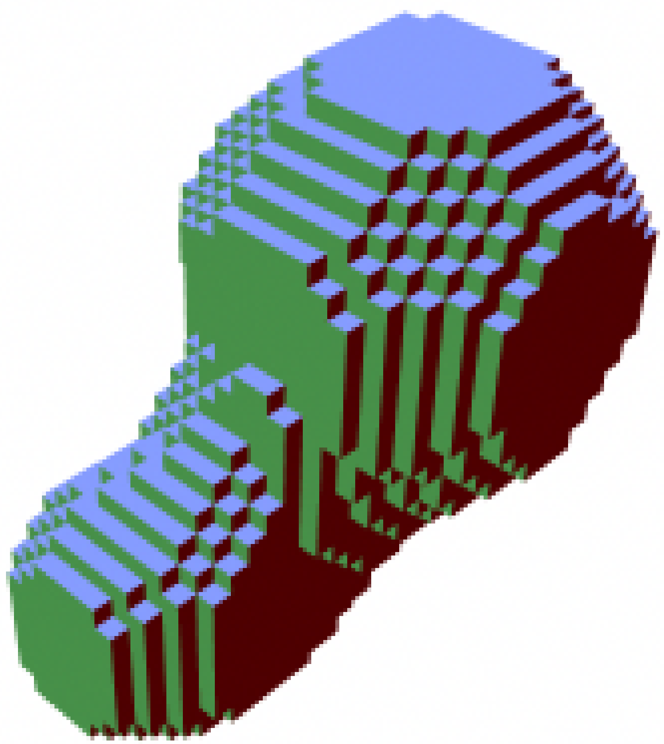

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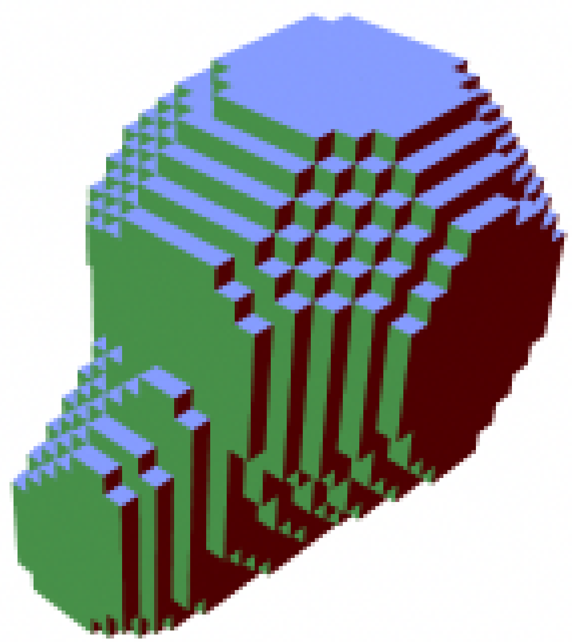

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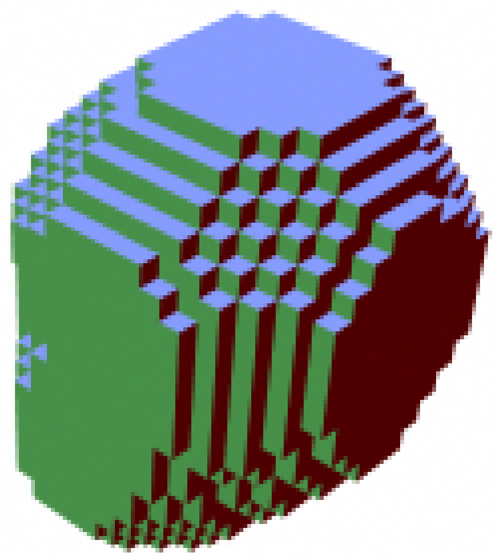
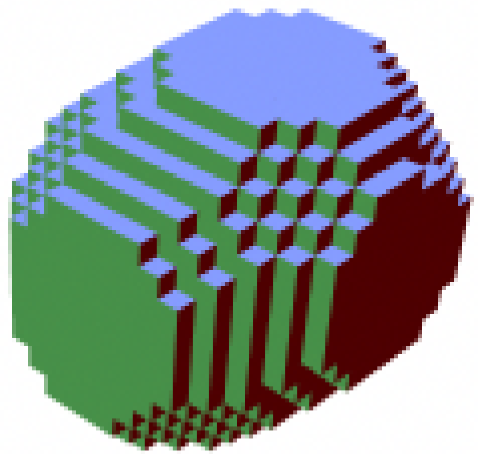


Integral and differential approaches:

Gradient flows via discrete graph cuts

$$F(C) = \| C \|_{\varepsilon}$$

mean curvature motion in 3D



Integral and differential approaches:

Gradient flows via discrete graph cuts

16-grid

mean curvature motion

Boykov, Kolmogorov, Cremers, Delong, ECCV '06

Integral and differential approaches:

Earlier discrete methods for local optima

- Banded graph cuts [Xu et al., CVPR 03]
  - binary 0-1 metric on the space of contours
    - thresholding Hausdorff distance between contours
  - jerky motion
  - produces "erosion" in case of the *sausage* example
  - $r(t) = const\text{-}t$ in case of a *collapsing circle* example
- DP-snakes [Amini et al., PAMI 1990]
  - Explicit boundary representation
    - constrained topology, non-geometric energy
  - Their method gives L1 metric on the space of contours
    - this is easy to correct based on insights in [BKCD, ECCV 2006]
  - 2D only

Integral and differential approaches:

PDE cuts, pluses and minuses

- Efficient binary search for *dt* (reuses residual graph)
  - No guessing for choosing time step is required
- No oscillatory motion, guaranteed energy decrease
- Does not need to estimate surface derivatives
- Should reset distance map to better approximate gradient flow in *L2* metric
- Can not produce arbitrarily small (sub-pixel) motion
- "Frying pan" artifact: small motion may be ignored if surface has large variation in curvature

Integral and differential approaches:

Summary

- Level-sets are based on ideas from **differential geometry**
  - sub-pixel accuracy, estimates derivatives
- Graph cuts use **integral geometry** to estimate length
  - no sub-pixel accuracy, but derivatives are unnecessary

- Level sets compute gradient flow by estimating local **differential motion** (speed) of contour points
  - derivatives (e.g curvature) are estimated at every point
- Discrete or continuous max-flow algorithms directly estimate **integral motion** of a contour as a whole.
  - no derivatives at contour points are estimated