ECCV 2006 tutorial on

Graph Cuts vs. Level Sets

part IV

Global vs. local optimisation algorithms

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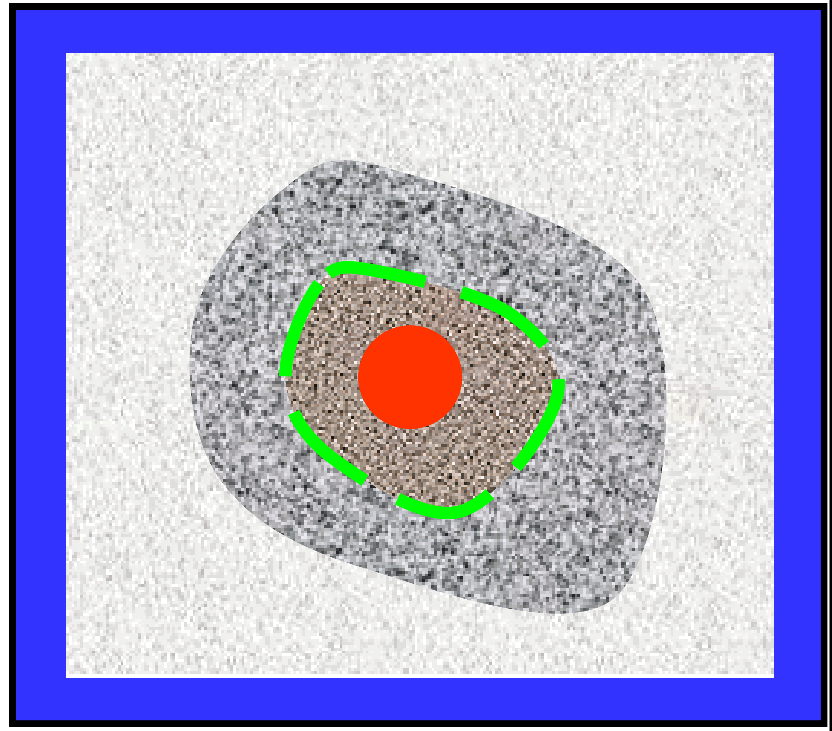
Global vs. local minima (Geodesic active contours)

- Geodesic active contours
  - Variational approach (e.g. level sets)
    - Gradient descent in the *space of contours*
    - Local minimum
    - Non-convex formulation?
  - Graph cuts (e.g. geo-cuts)
    - Same problem, global minimum
    - Convex formulation?



[Anonymous attendee of CVPR'05]:

How is it possible?

Global vs. local minima (Geodesic active contours)

- Geodesic active contours
  - Variational approach (e.g. level sets)
    - Gradient descent in the *space of contours*
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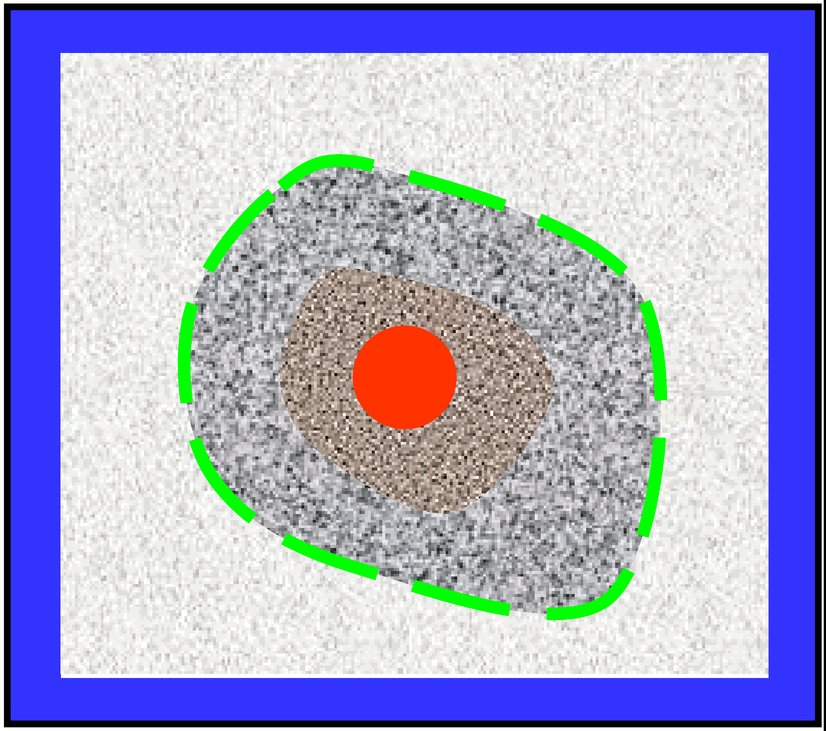


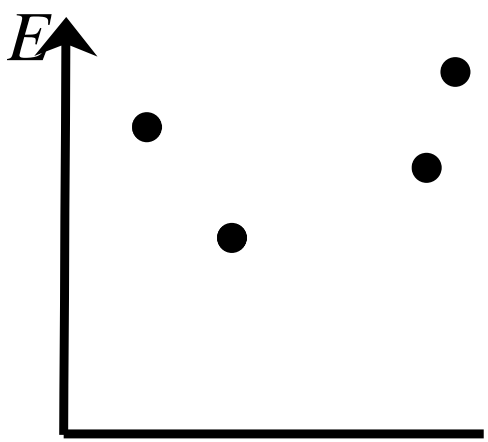
[Anonymous attendee of CVPR'05]:

How is it possible?

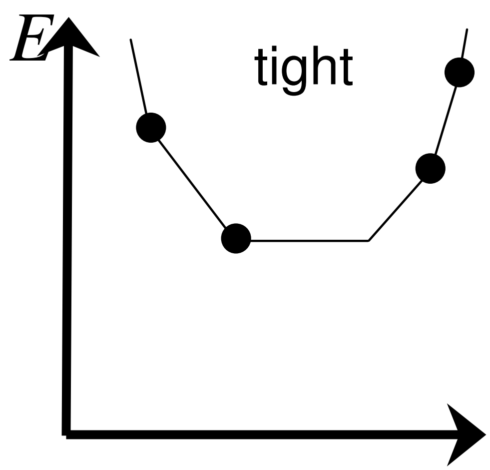
Graph cuts

- Function $E(\mathbf{x})$ of discrete variables: convexity not defined
- Extend the space of solutions: $x_p \in \{0,1\} \;\Rightarrow\; x_p \in [0,1]$
  - Allow *fractional* segmentations
- Extend energy: *linear programming (LP) relaxation*
  - Now convex problem!

submodular function ⇒ integer solution



Energy function
with discrete variables



LP relaxation

Graph cuts

- Function $E(\mathbf{x})$ of discrete variables: convexity not defined
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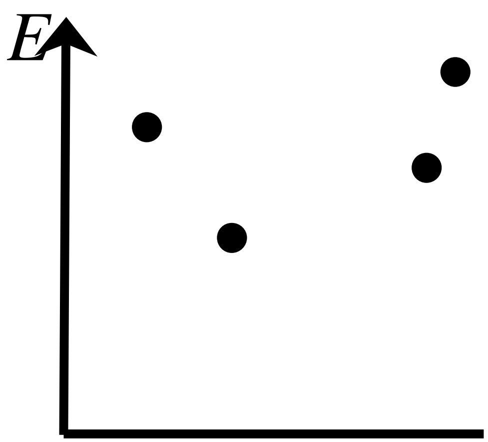
non-submodular function ⇒ fractional solution (in general)



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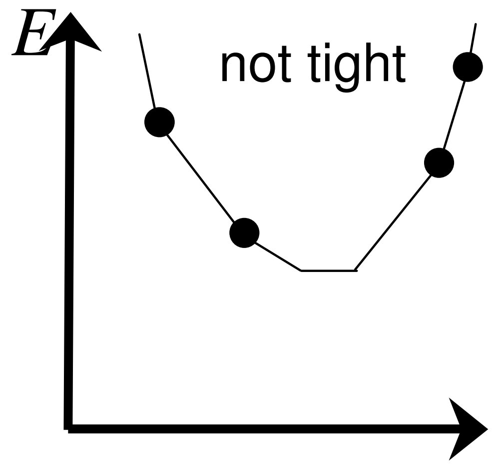
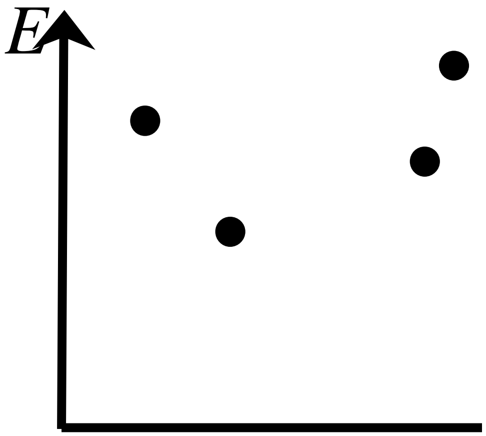


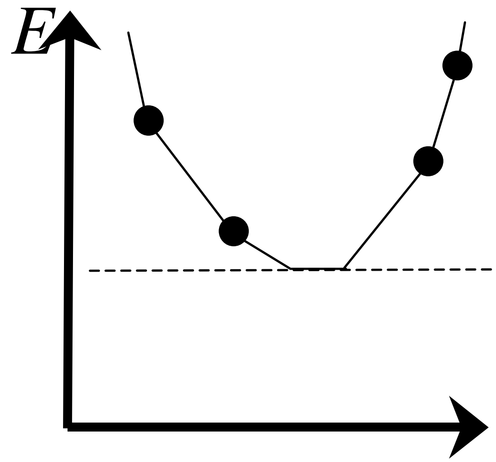
LP relaxation

Solving LP relaxation

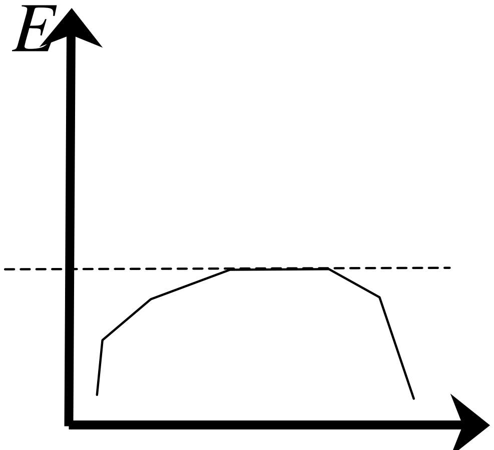
- Too large for general purpose LP solvers (e.g. interior point methods)
- Solve *dual* problem instead of *primal*:
  - Formulate lower bound on the energy
  - Maximize this bound
  - When done, solves primal problem (LP relaxation)
- Two different ways to formulate lower bound
  - Part A: Via *posiforms* => maxflow algorithm (for binary variables)
  - Part B: Via *convex combination of trees* => tree-reweighted message passing



Energy function with discrete variables



LP relaxation

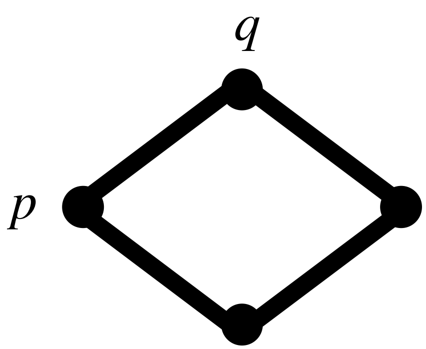


Lower bound on the energy function

Notation and Preliminaries

Energy function

$$E(\mathbf{x} \mid \theta) = \theta_{const} + \underbrace{\sum_{p} \theta_p(x_p)}_{\text{unary terms (data)}} + \underbrace{\sum_{p,q} \theta_{pq}(x_p, x_q)}_{\text{pairwise terms (coherence)}}$$



- x_p are discrete variables (for example, $x_p \in \{0,1\}$)
- $\theta_p(\cdot)$ are unary potentials
- $\theta_{pq}(\cdot,\cdot)$ are pairwise potentials

LP relaxation

- [Schlesinger'76,Koster *et al.*'98,Chekuri *et al.*'00,Wainwright *et al.*'03]

- Introduce indicator variables $x_{p;i}$, $x_{pq;ij}$

$$\sum_{p,i} \theta_p(i) x_{p;i} + \sum_{p,q,i,j} \theta_{pq}(i,j) x_{pq;ij} \rightarrow \min$$

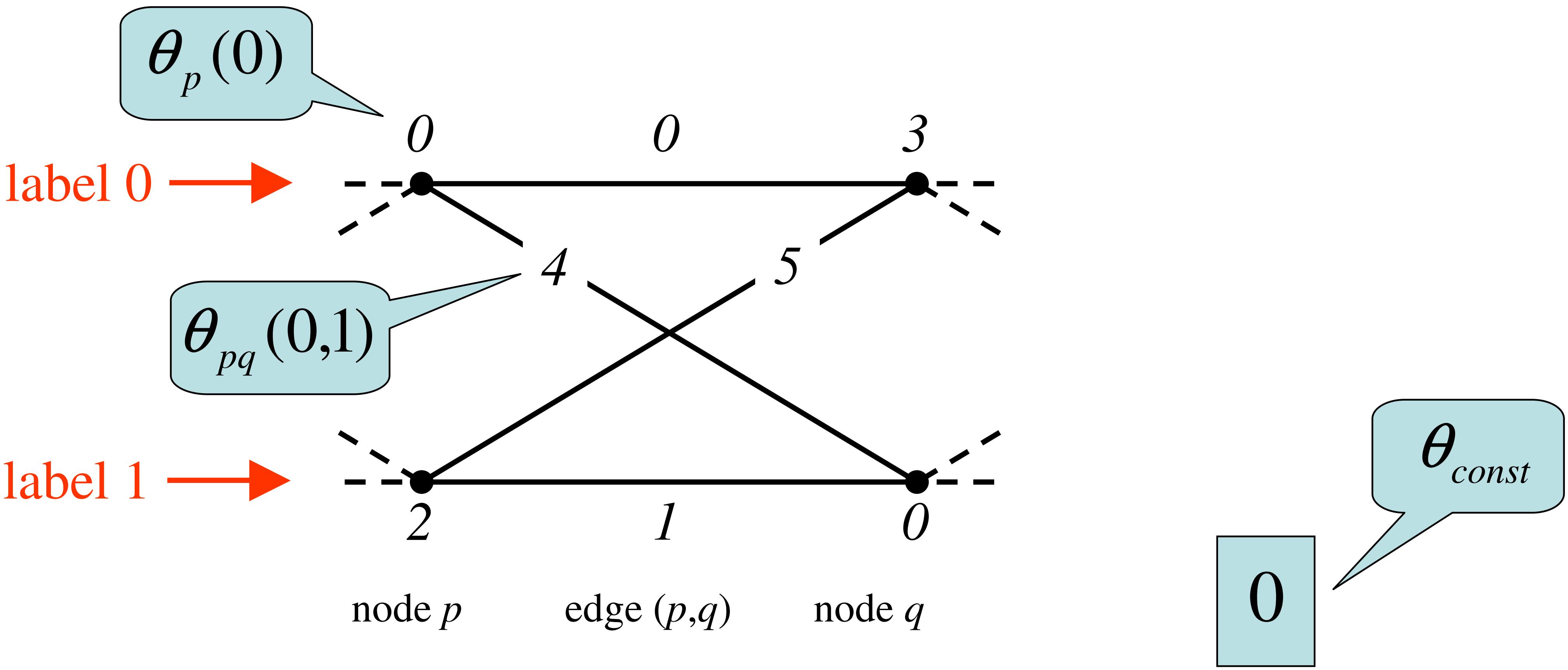
$$\begin{cases} \sum_j x_{pq;ij} = x_{p;i} \\ \sum_i x_{p;i} = 1 \\ x_{pq;ij} \in \{0,1\} \end{cases}$$

relaxation

$$x_{pq;ij} \in [0,1]$$

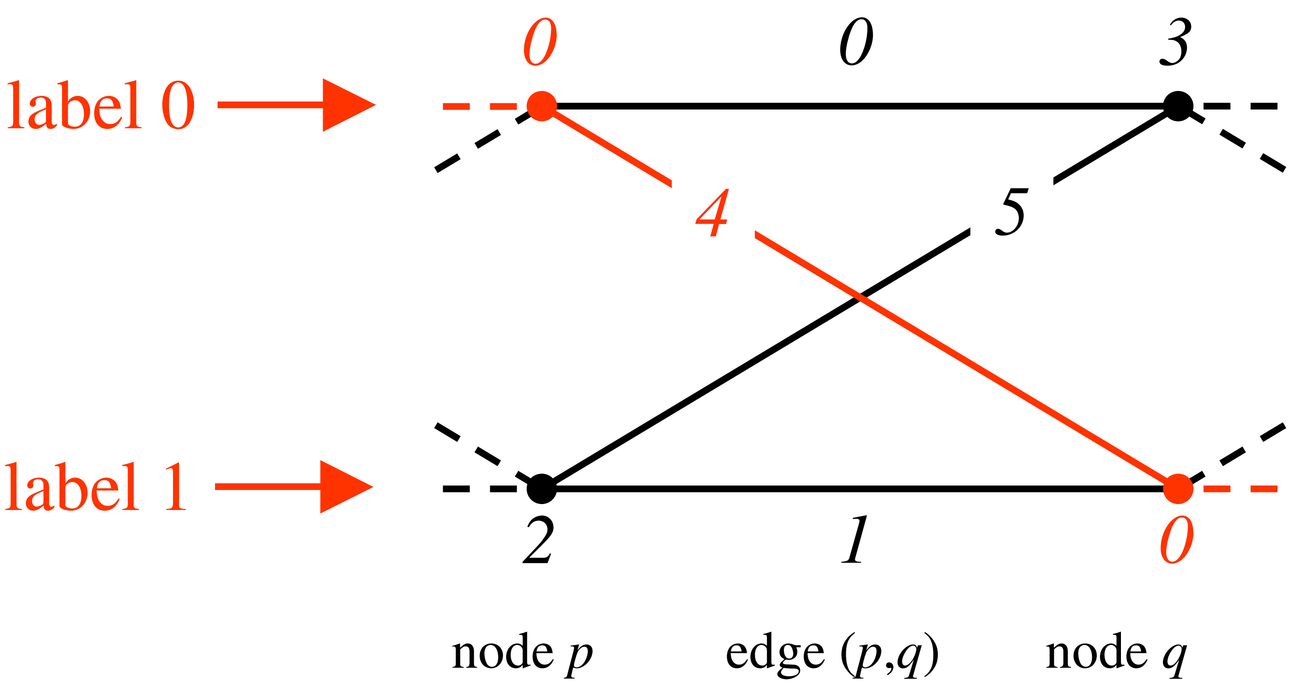
Energy function - visualisation

$$E(\mathbf{x}\,|\,\theta) = \theta_{const} + \sum_{p} \theta_p(x_p) + \sum_{p,q} \theta_{pq}(x_p, x_q)$$



Energy function - visualisation

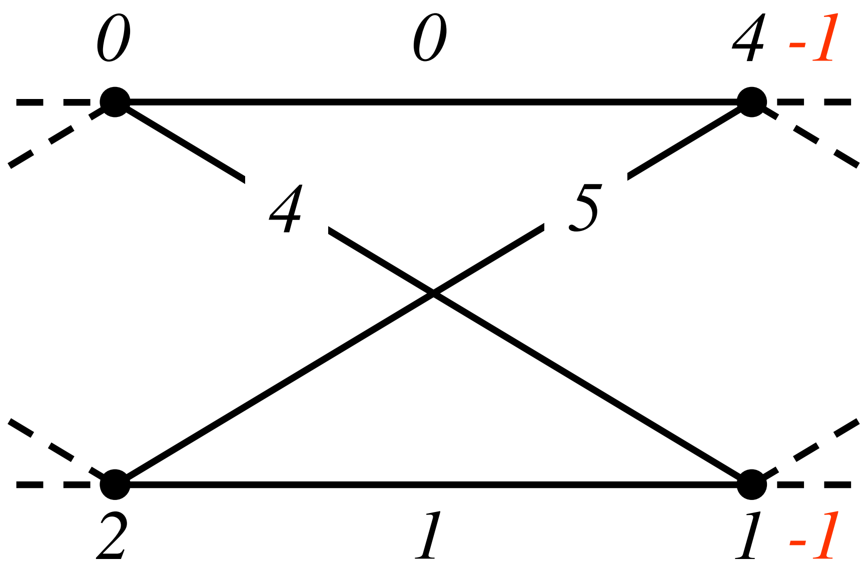
$$E(\mathbf{x} \mid \theta) = \theta_{const} + \sum_{p} \theta_p(x_p) + \sum_{p,q} \theta_{pq}(x_p, x_q)$$



$\theta =$ vector of all parameters

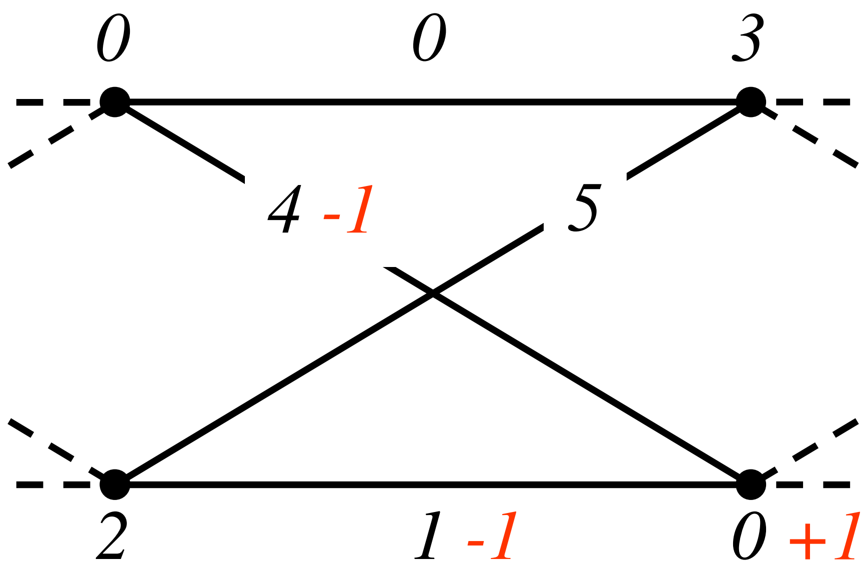
0

Reparameterisation



0 + 1

Reparameterisation



1

- **Definition.** θ' is a reparameterisation of θ $(\theta' \equiv \theta)$ if they define the same energy:

$$E(\mathbf{x} \mid \theta') = E(\mathbf{x} \mid \theta) \quad \text{for any } \mathbf{x}$$

- Maxflow, BP and TRW perform reparameterisations

Part A: Lower bound via posiforms

($\Rightarrow$ maxflow algorithm)

Lower bound via posiforms [Hammer, Hansen, Simeone'84]

$$E(\mathbf{x} \mid \theta) = \theta_{const} + \underbrace{\sum_{p} \theta_p(x_p) + \sum_{p,q} \theta_{pq}(x_p, x_q)}_{\text{non-negative}}$$

maximize (θ_{const})

θ_{const} - lower bound on the energy:

$$E(\mathbf{x} \mid \theta) \geq \theta_{const} \qquad \forall \mathbf{x}$$

Outline of part A

- Posiform maximisation: algorithm?

- Binary variables, *submodular* functions
  - Reduction to maxflow
  - Global minimum of the energy

- Binary variables, *non-submodular* functions
  - Reduction to maxflow
    - More complicated graph
  - *Part* of optimal solution

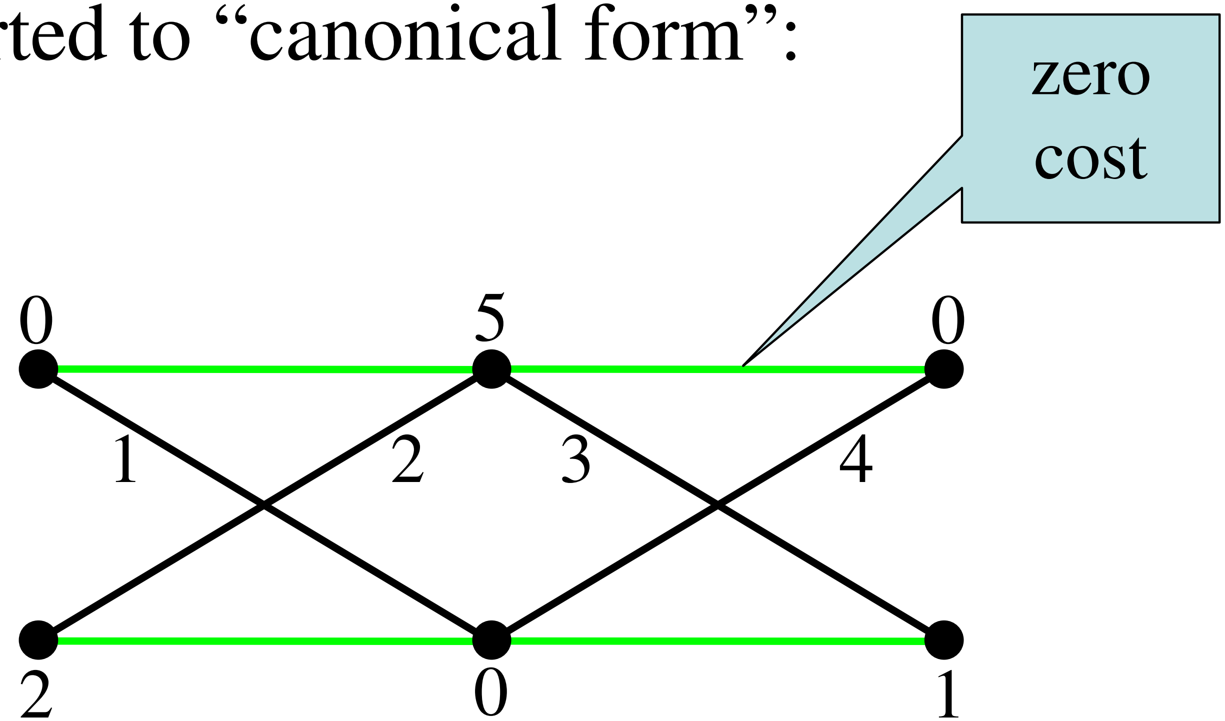
Binary variables, submodular functions

Submodularity and canonical form

- Definition: *E* is *submodular* if every pairwise term satisfies

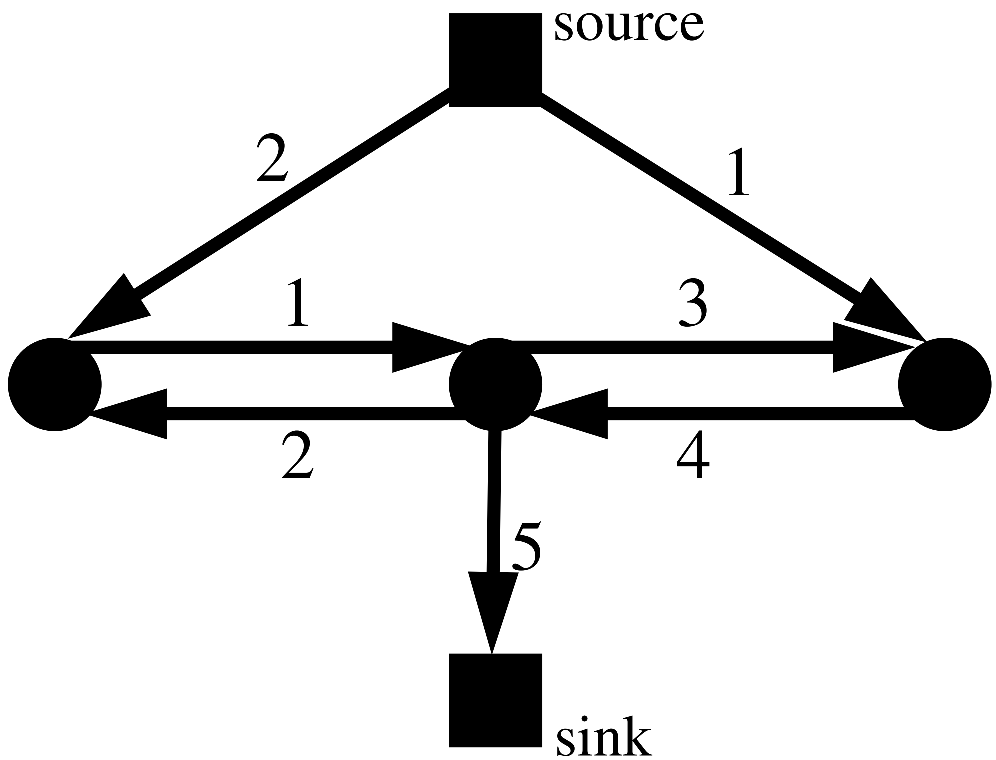
$$\theta_{pq}(0,0) + \theta_{pq}(1,1) \leq \theta_{pq}(0,1) + \theta_{pq}(1,0)$$

- Can be converted to "canonical form":



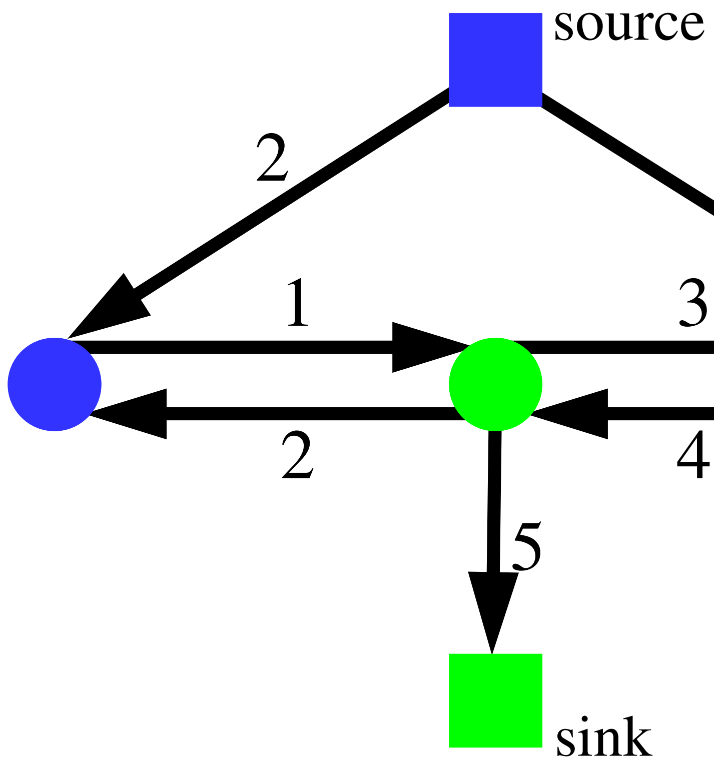
Overview of min cut/max flow

Min Cut problem



Directed weighted graph

Min Cut problem

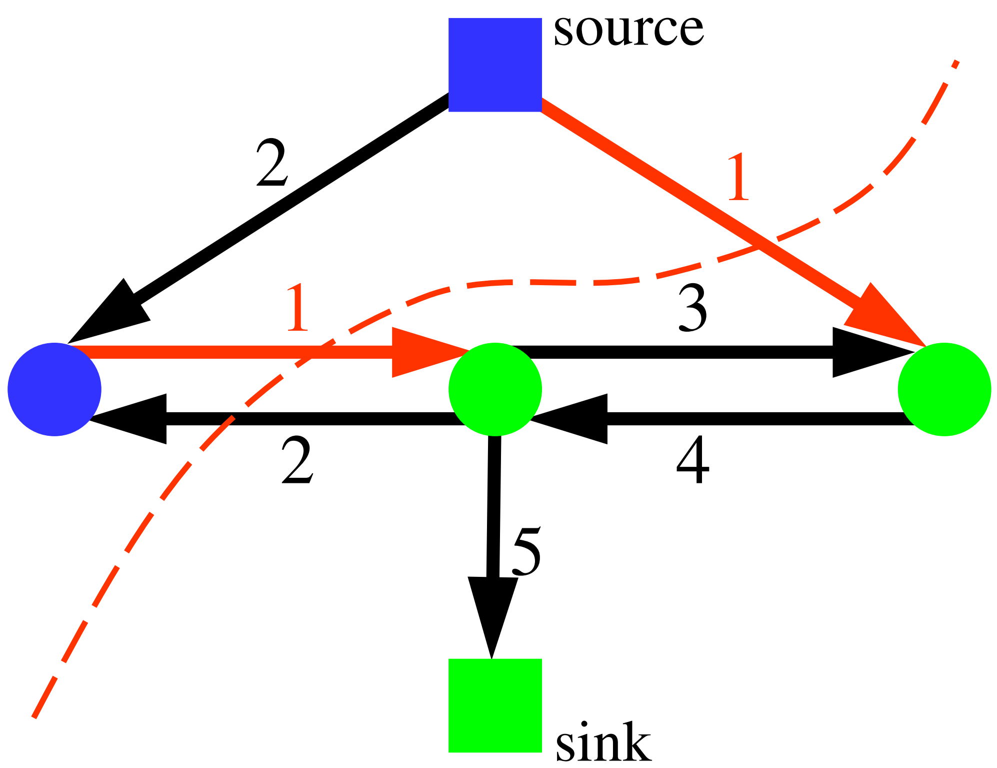


Cut:

S = {source, node 1}

T = {sink, node 2, node 3}

Min Cut problem



Cut:

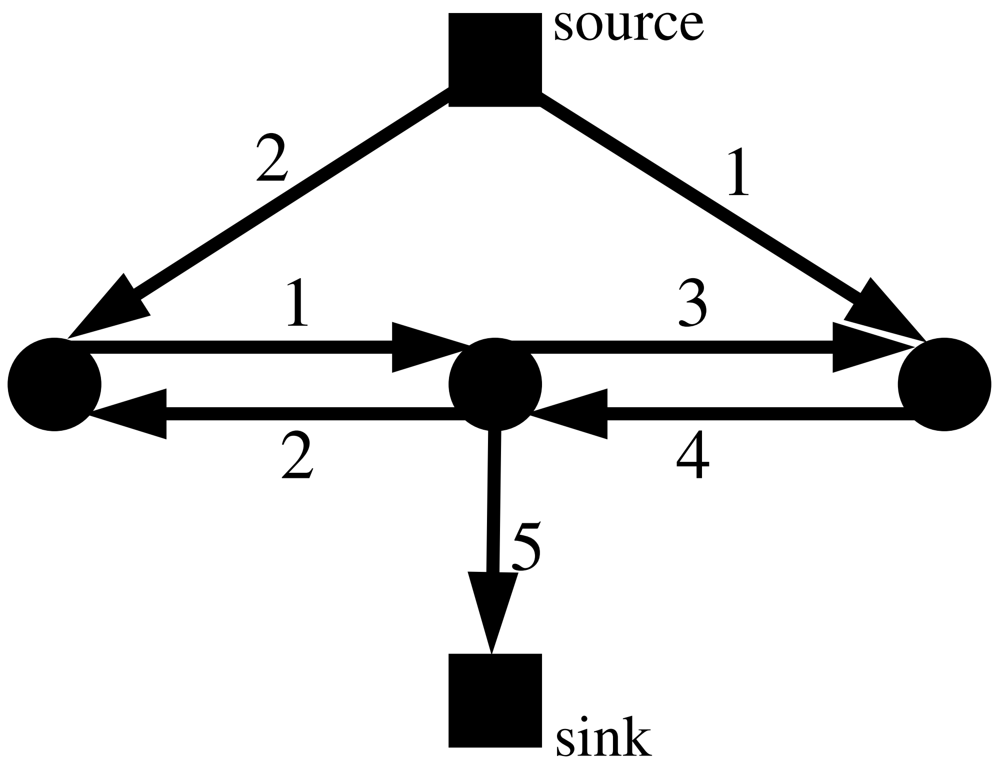
S = {source, node 1}

T = {sink, node 2, node 3}

Cost(S,T) = 1 + 1 = 2

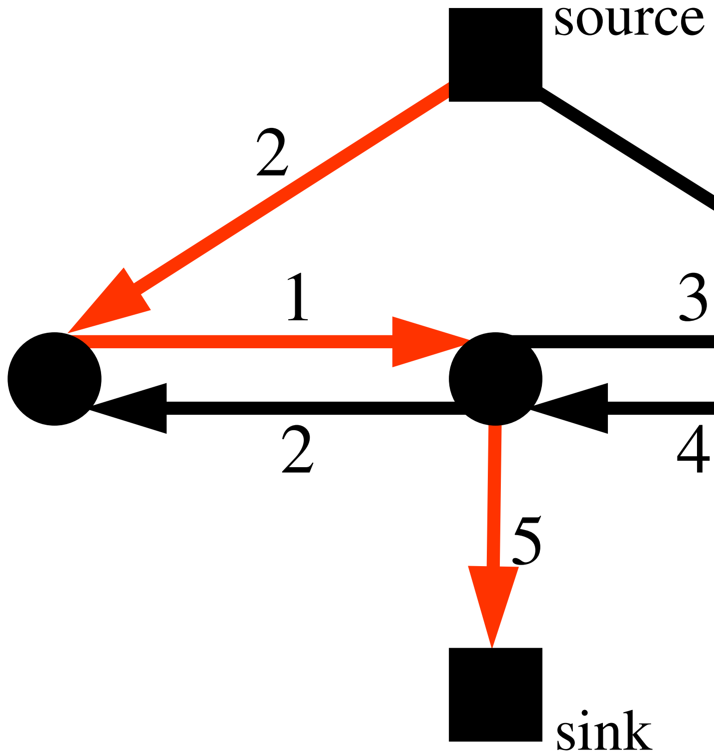
- Task: Compute cut with minimum cost

Maxflow algorithm



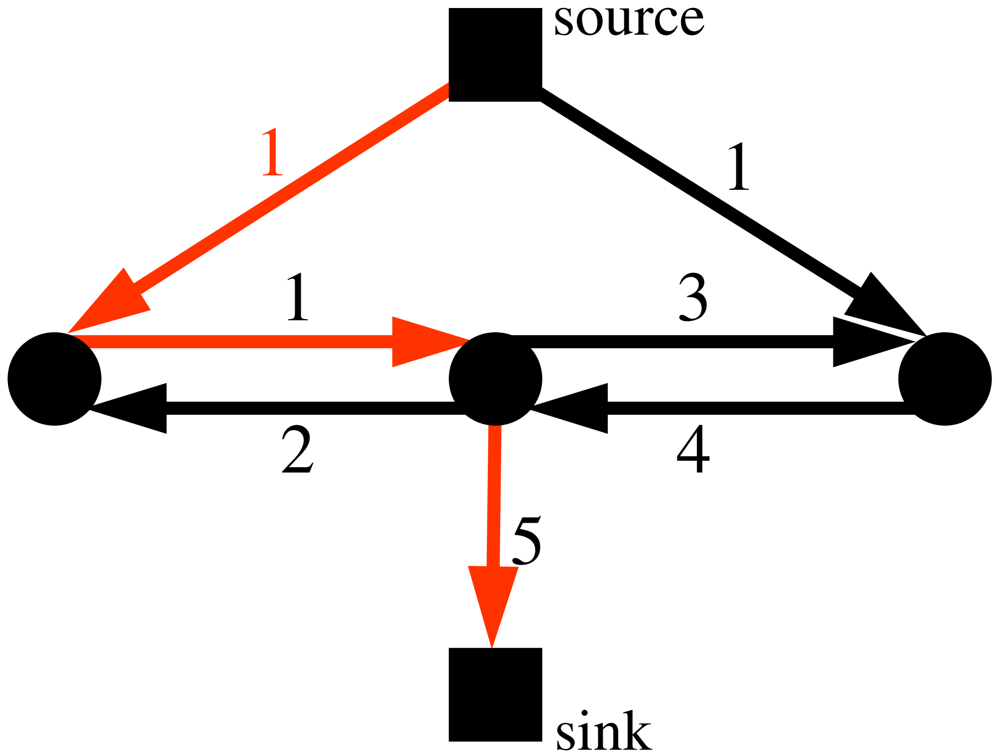
value(flow)=0

Maxflow algorithm



value(flow)=0

Maxflow algorithm



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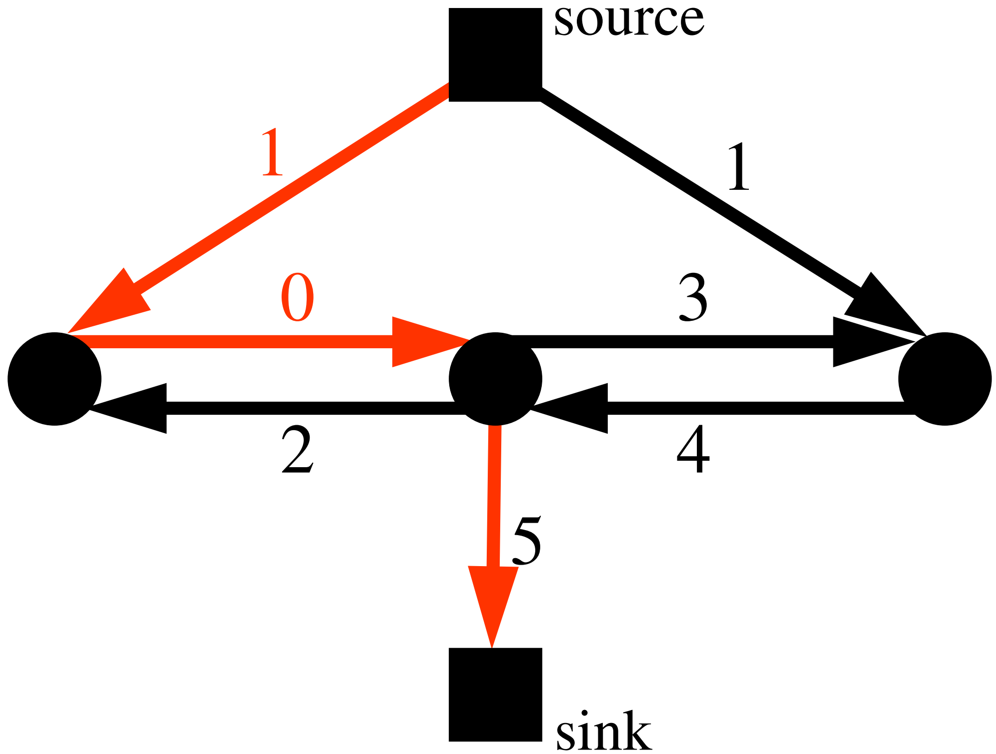
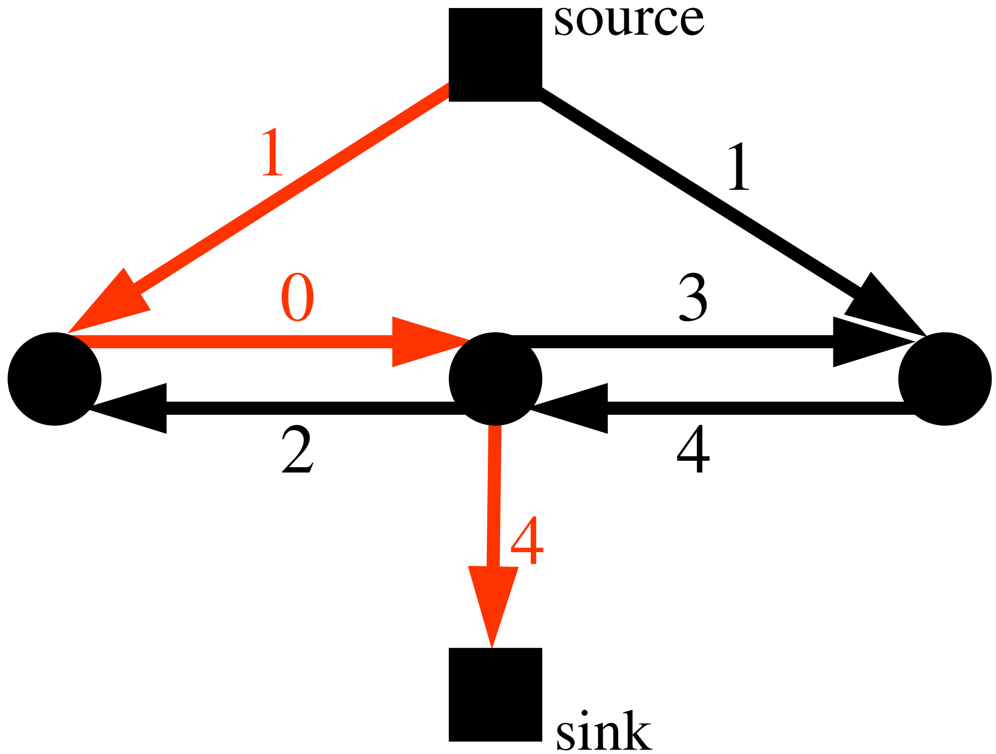


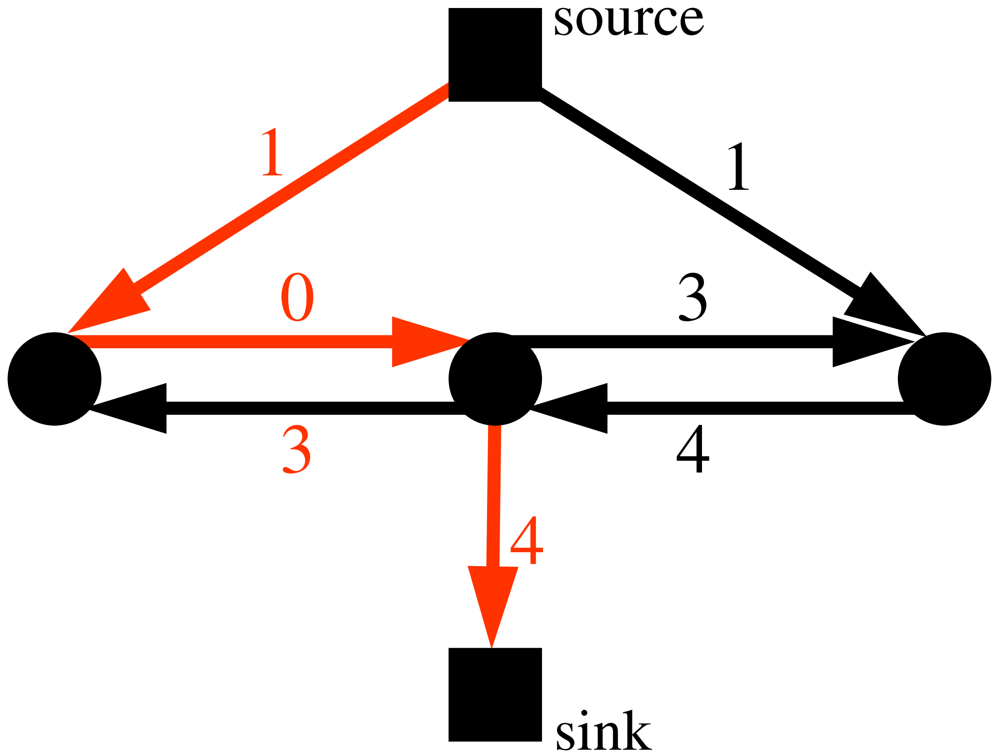
value(flow)=0

Maxflow algorithm



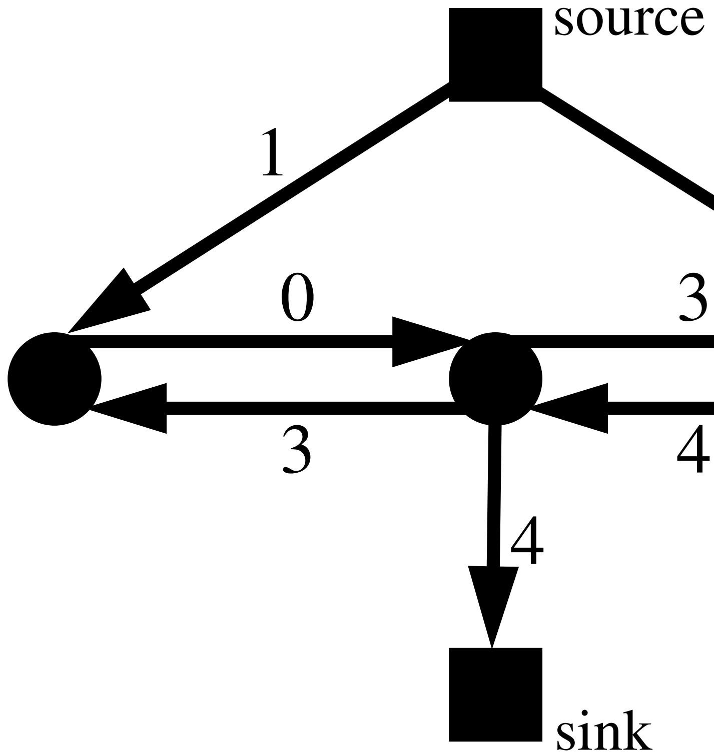
value(flow)=1

Maxflow algorithm



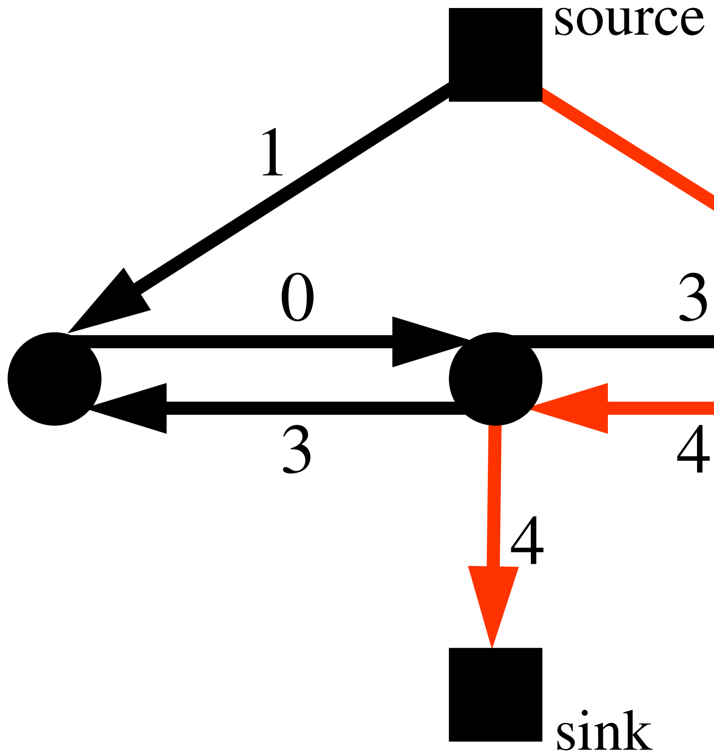
value(flow)=1

Maxflow algorithm



value(flow)=1

Maxflow algorithm



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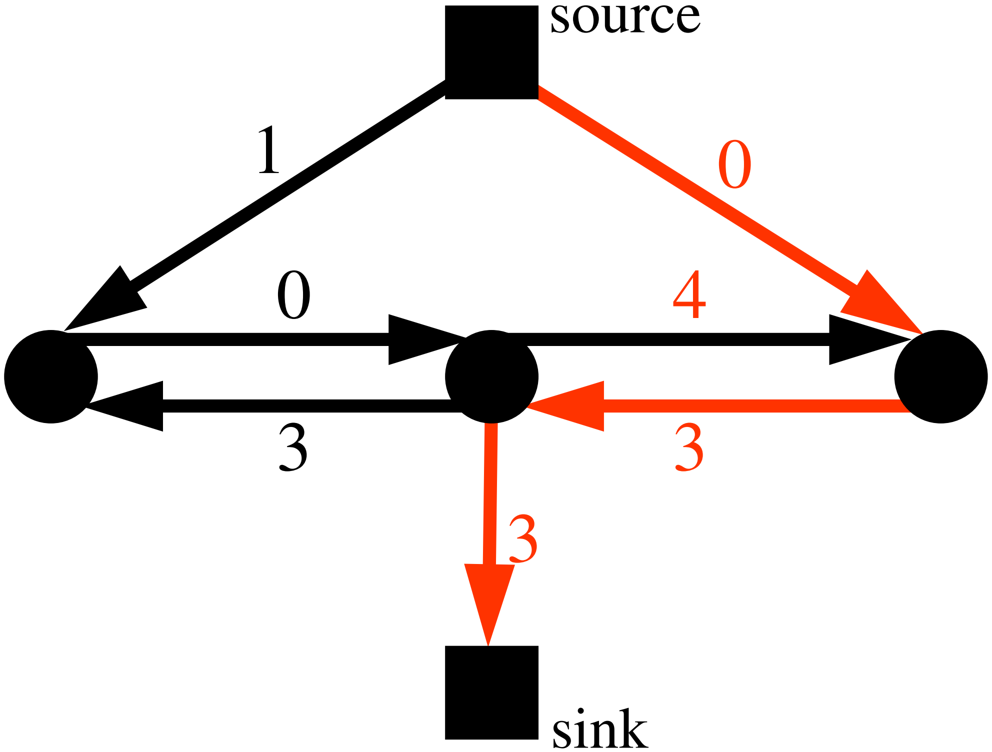
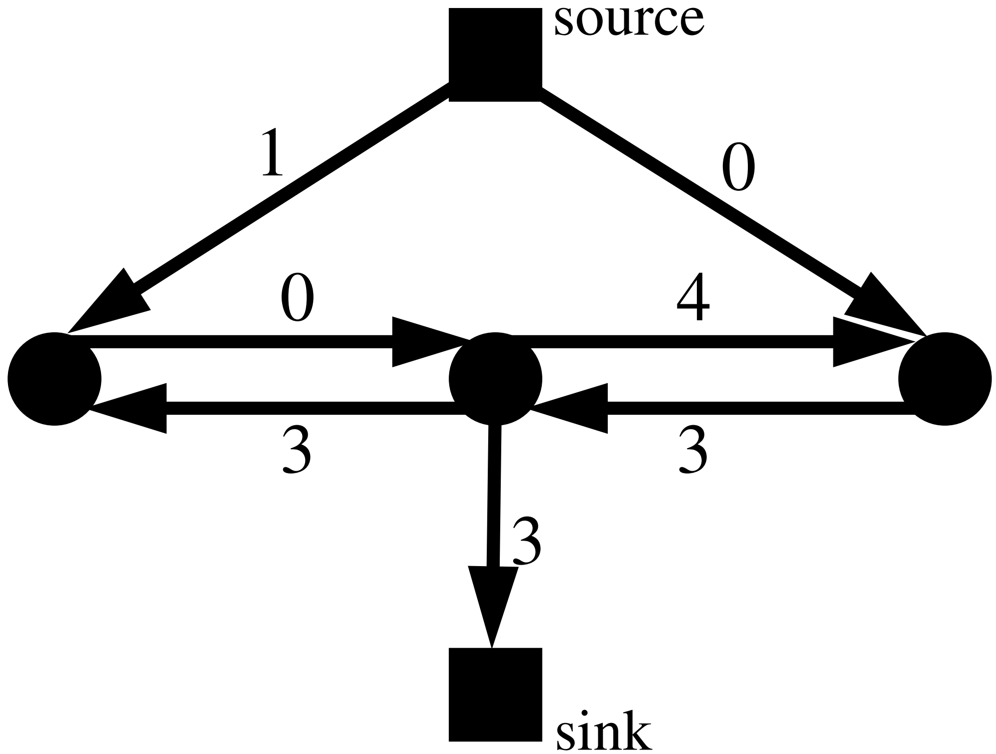
Maxflow algorithm



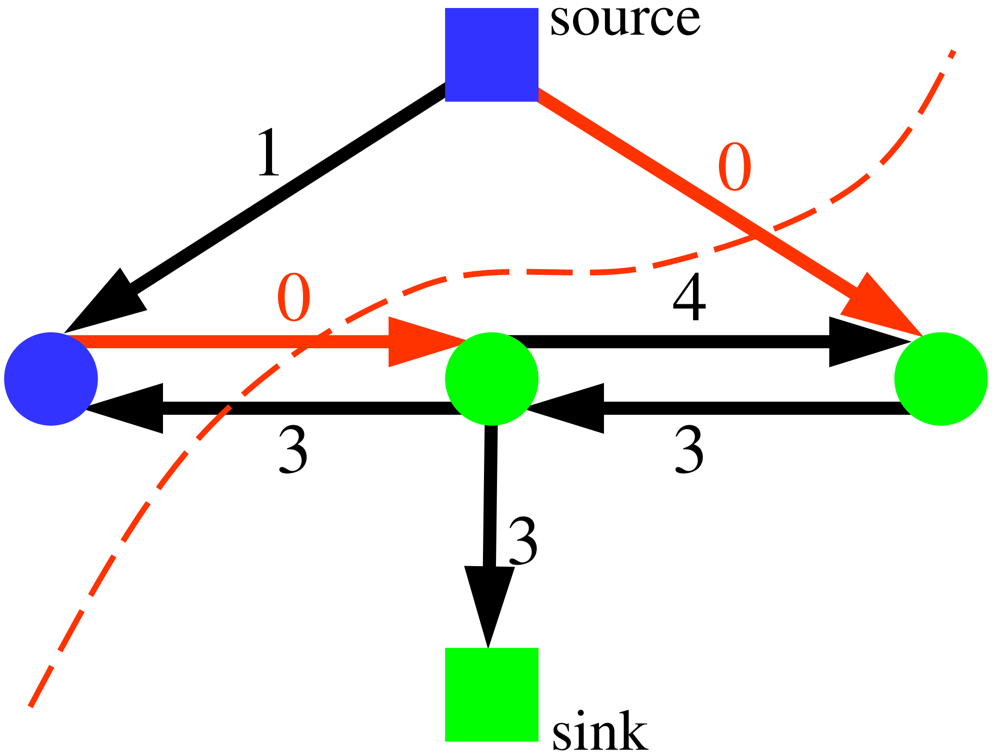
value(flow)=2

Maxflow algorithm



value(flow)=2

Maxflow algorithm

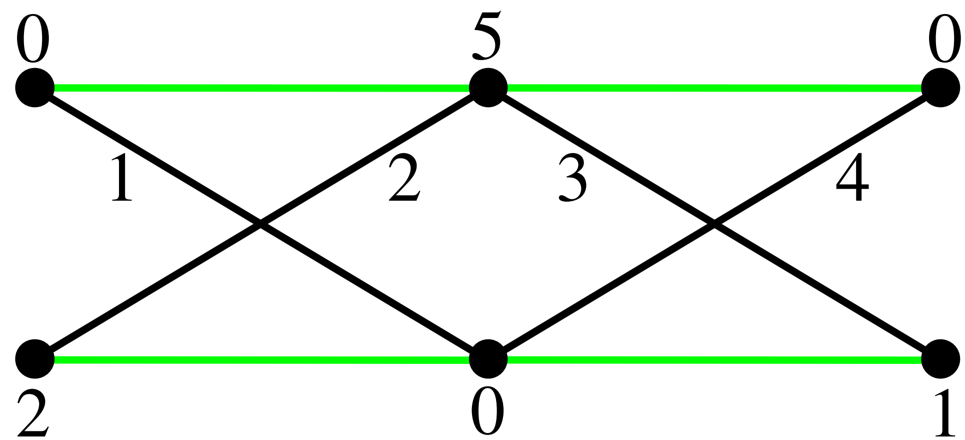


value(flow)=2

Binary variables,
non-submodular functions

Reduction to maxflow

Maxflow algorithm and reparameterisation



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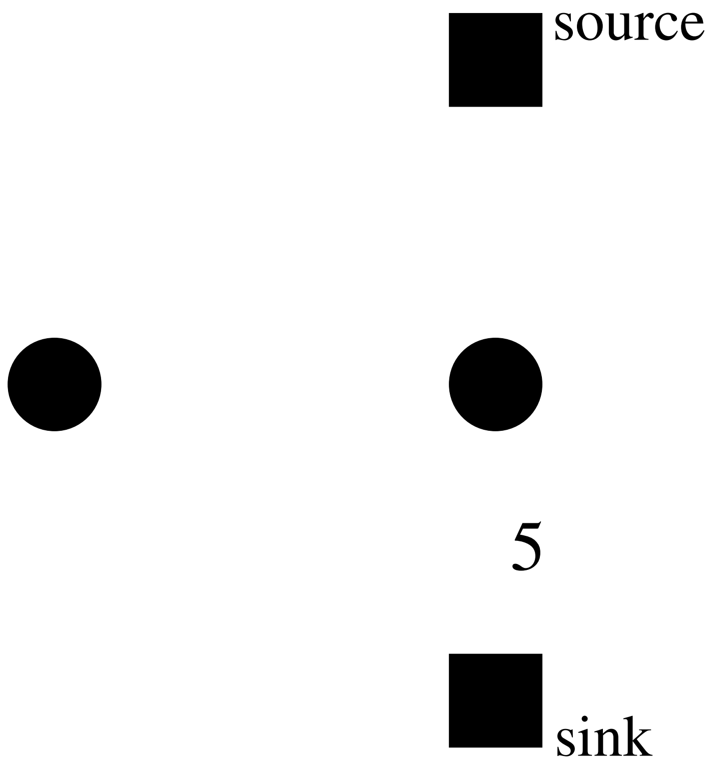

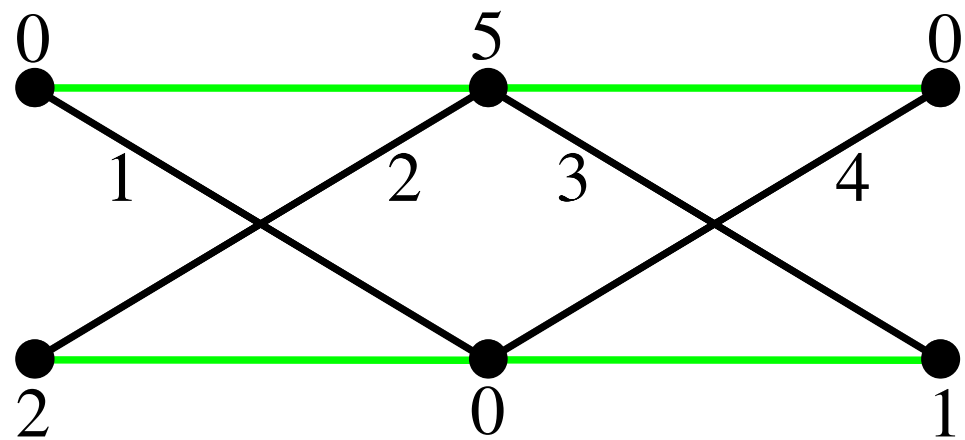

Maxflow algorithm
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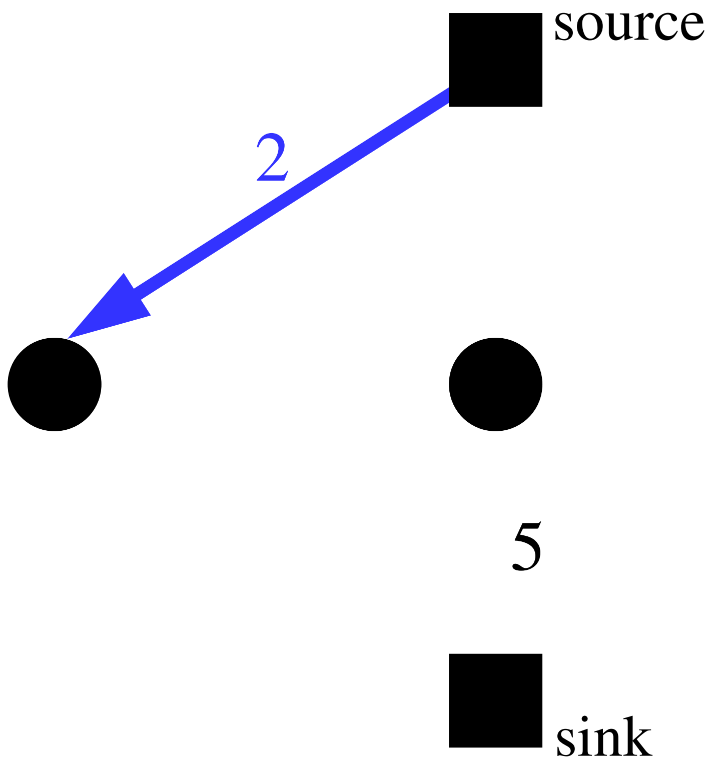

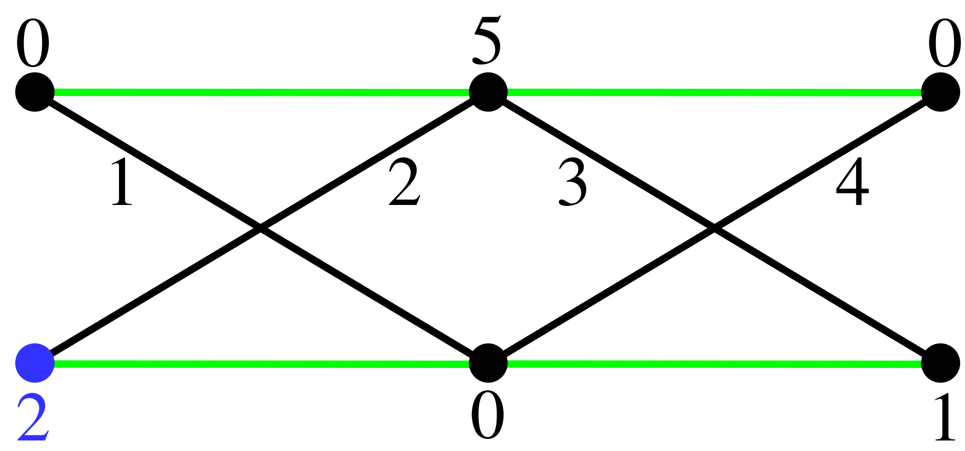

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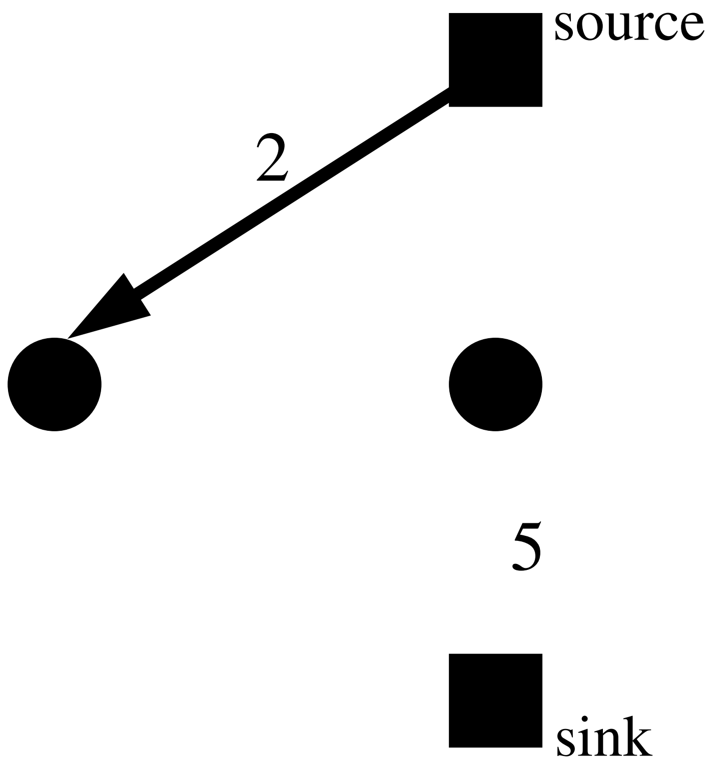

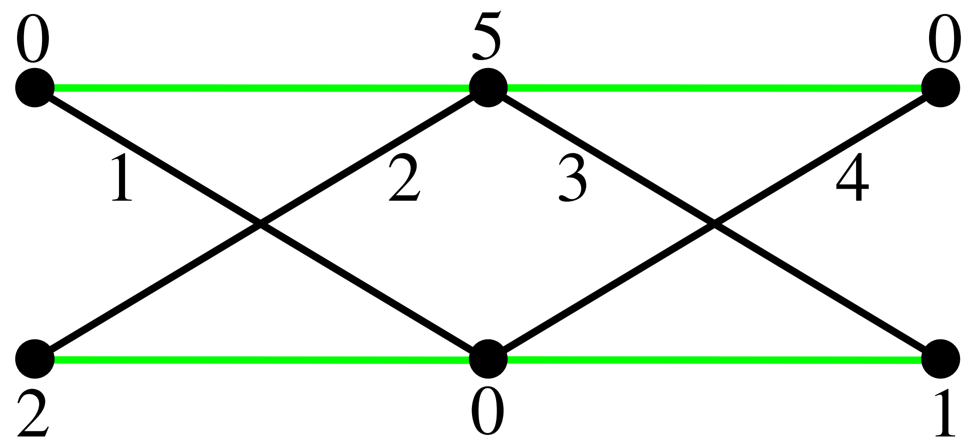

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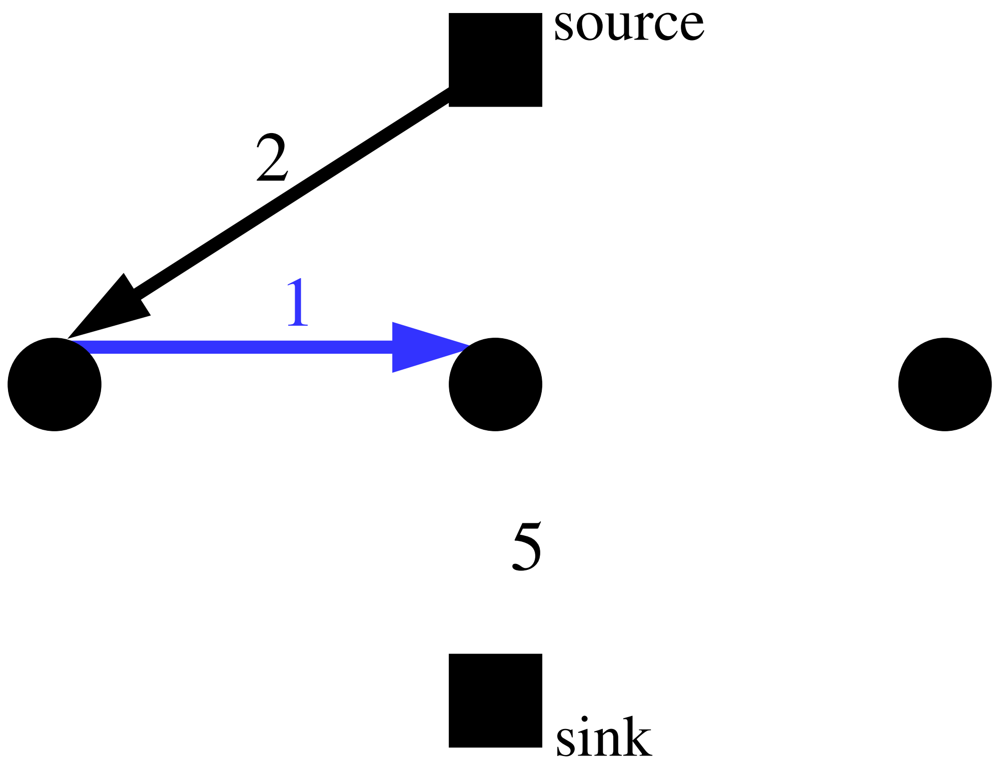

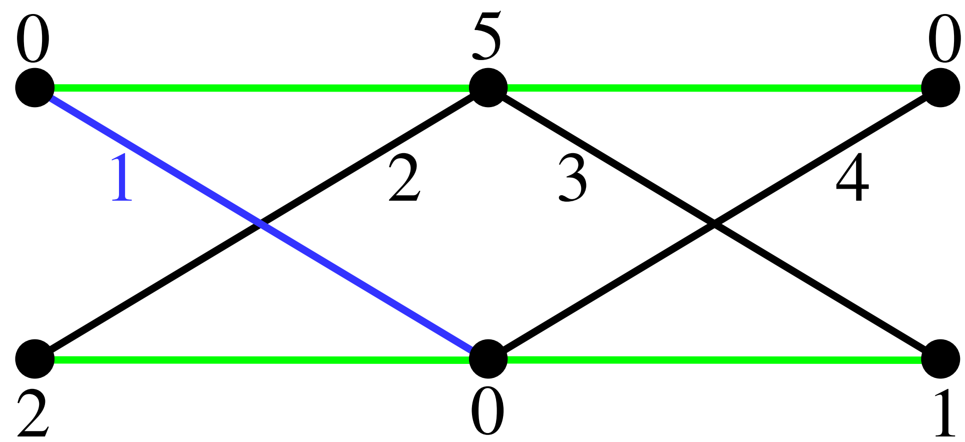

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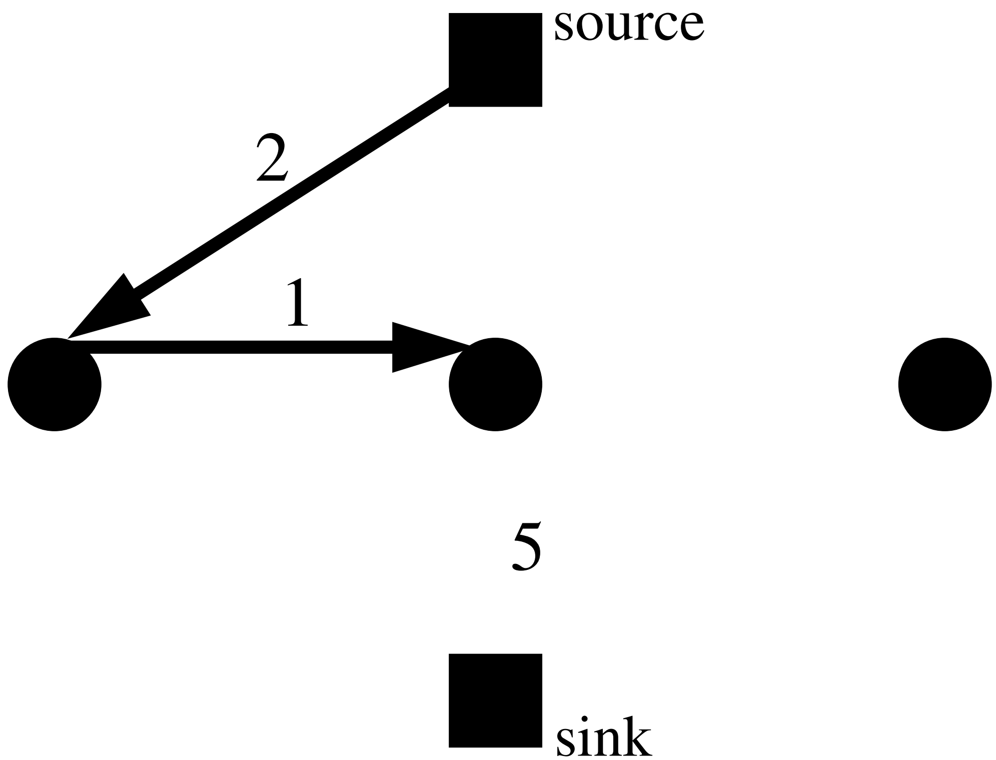

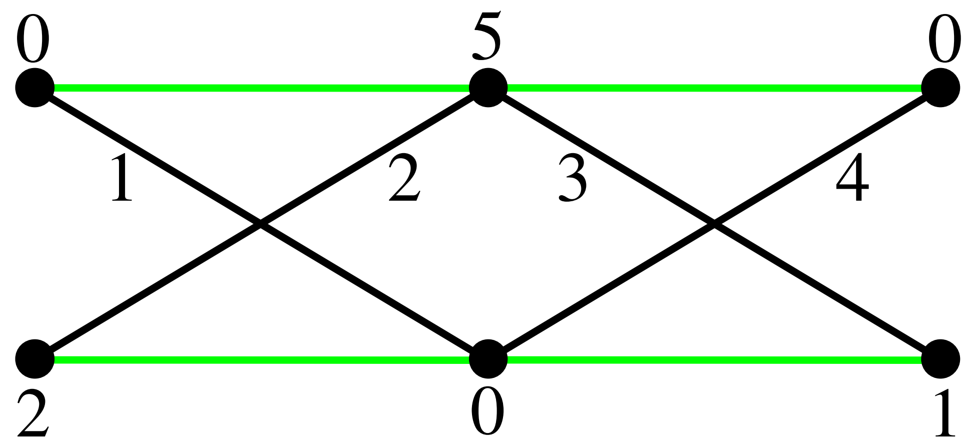

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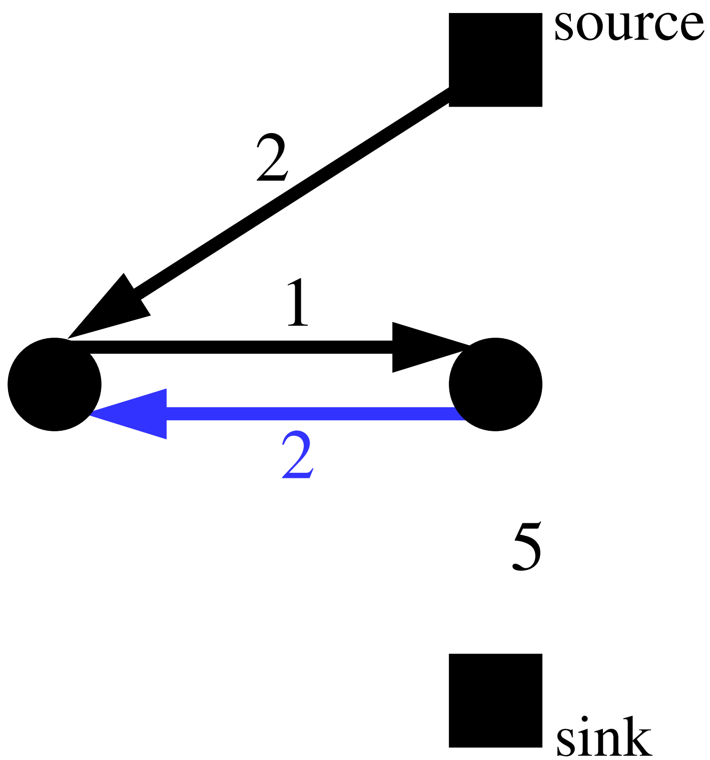

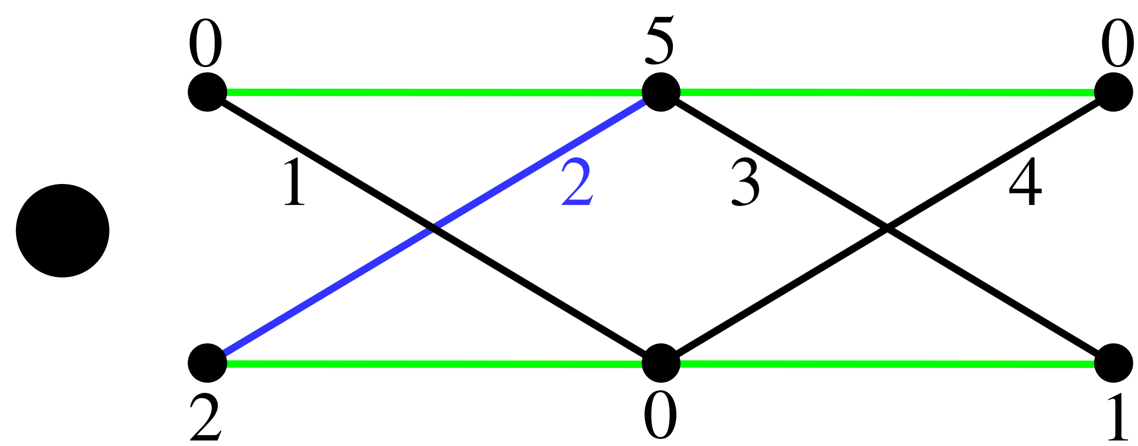

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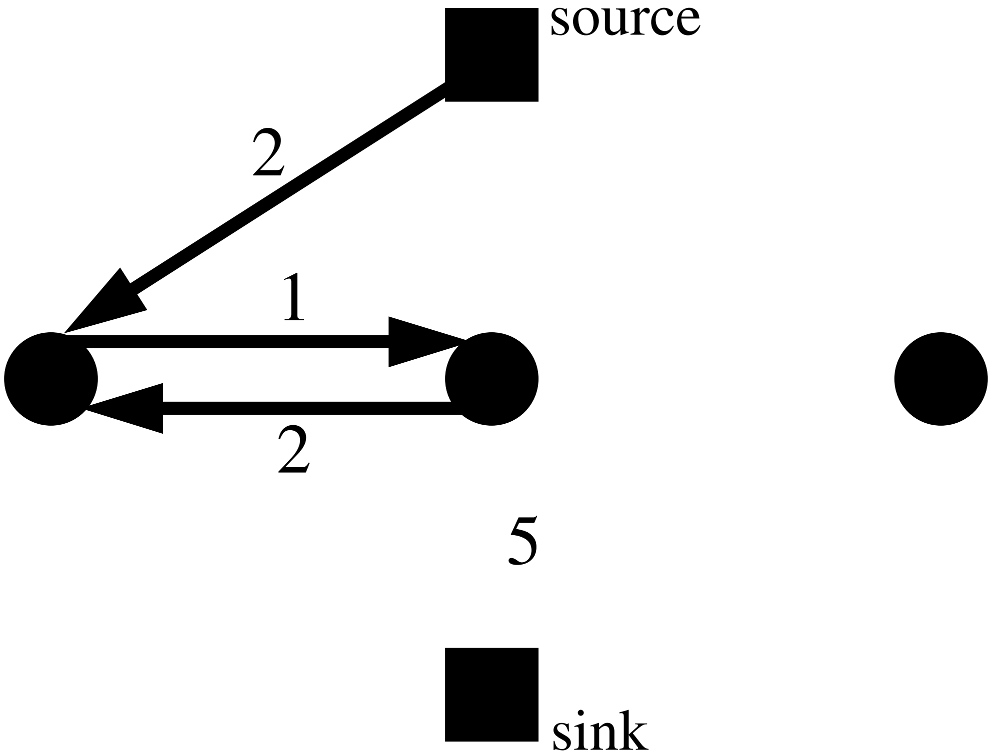

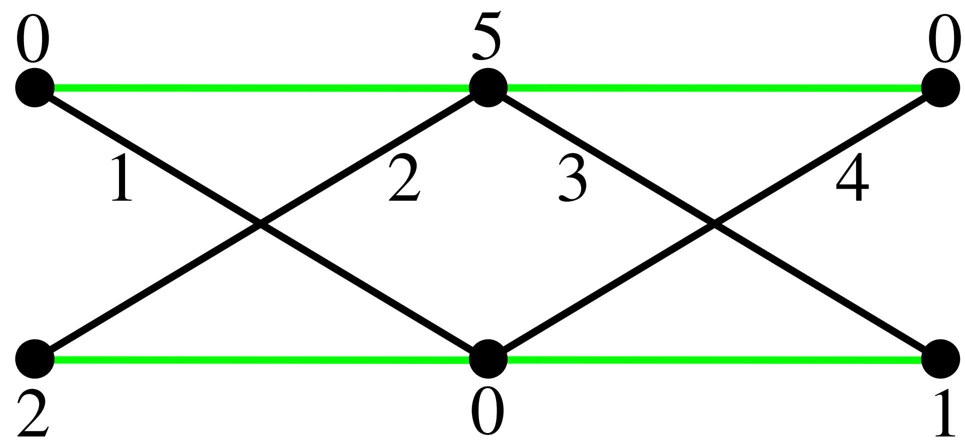

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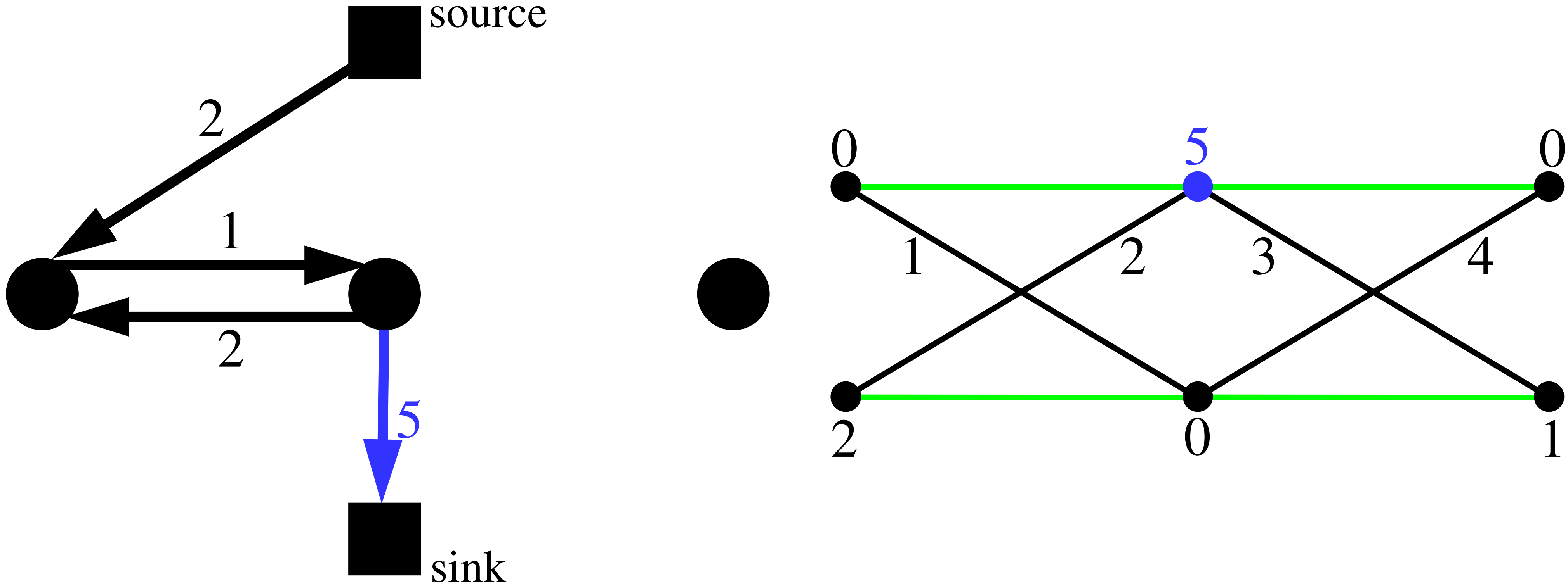

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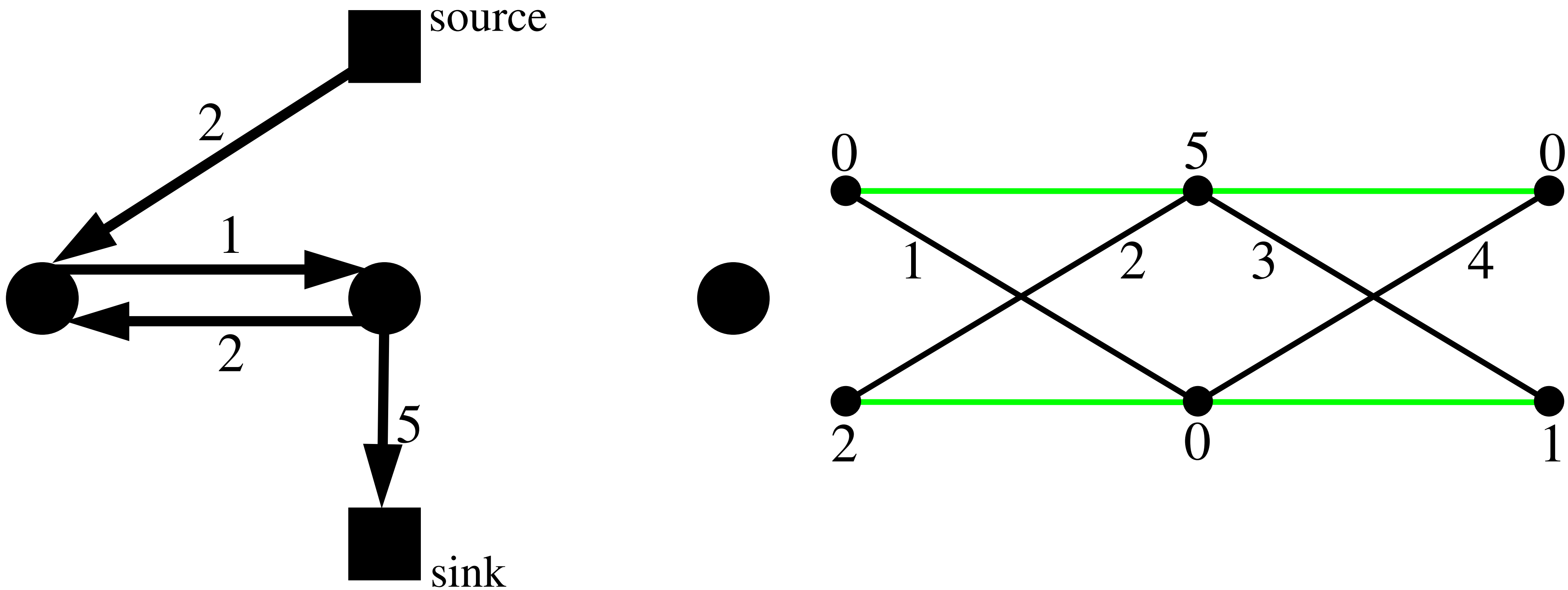

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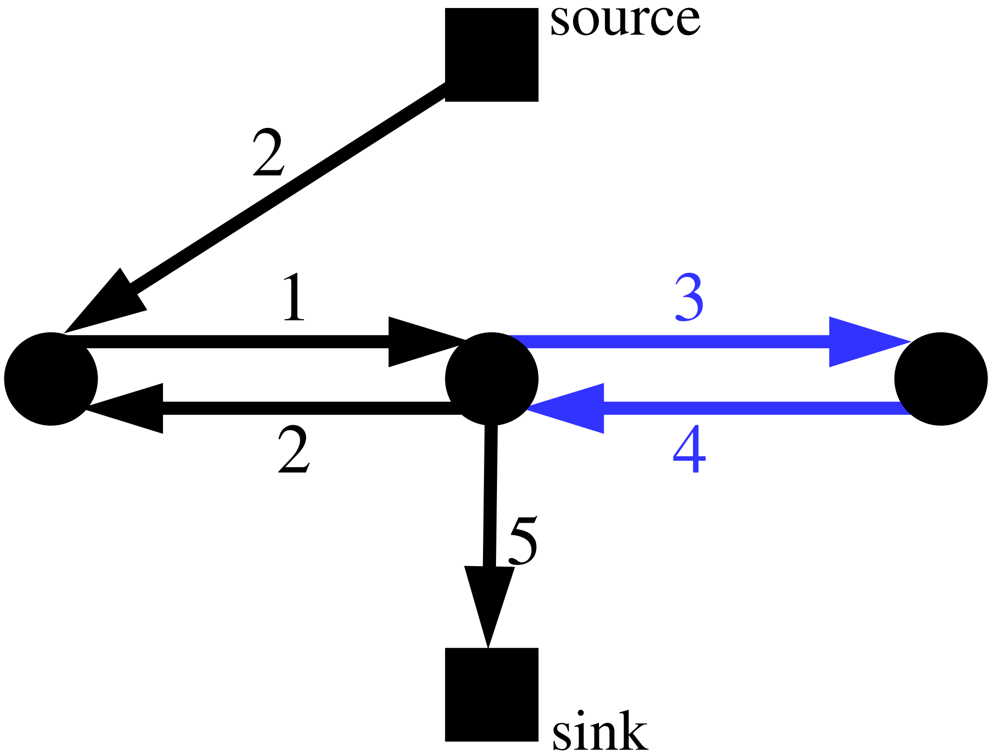

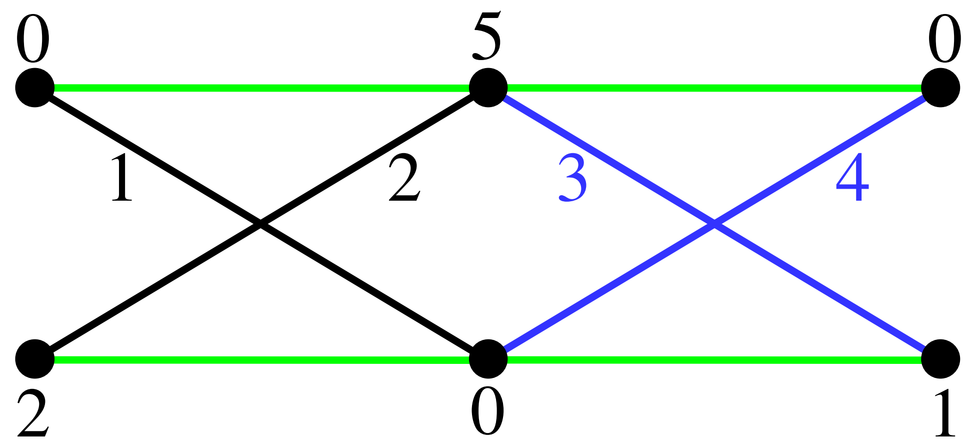

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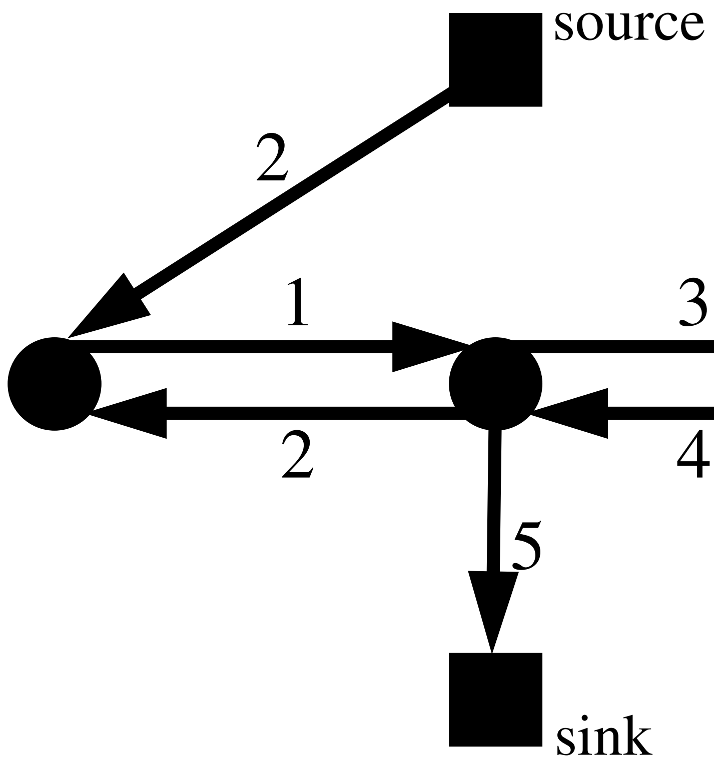

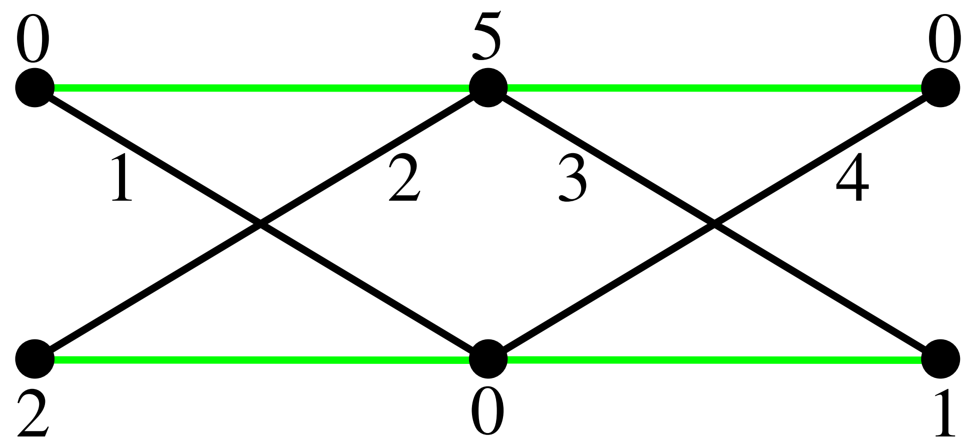

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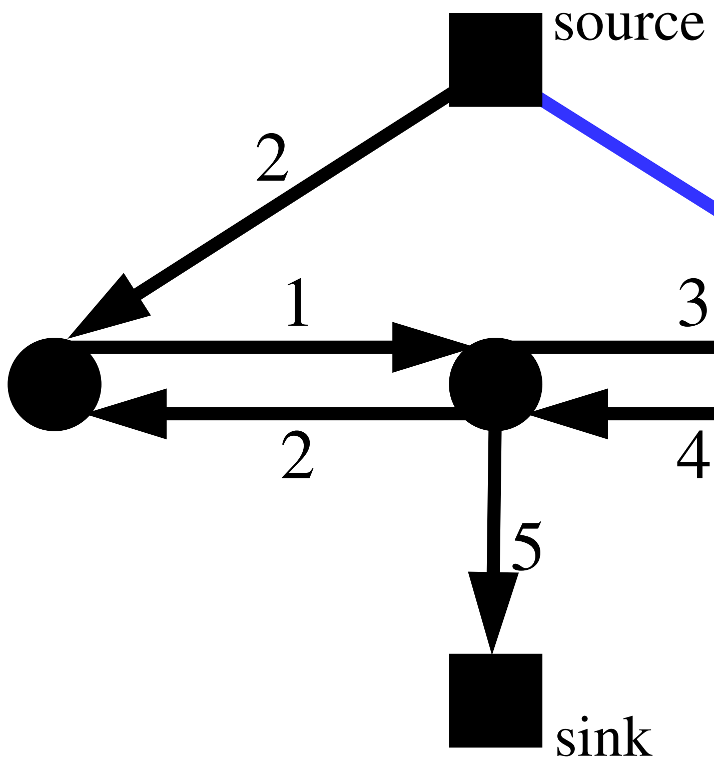

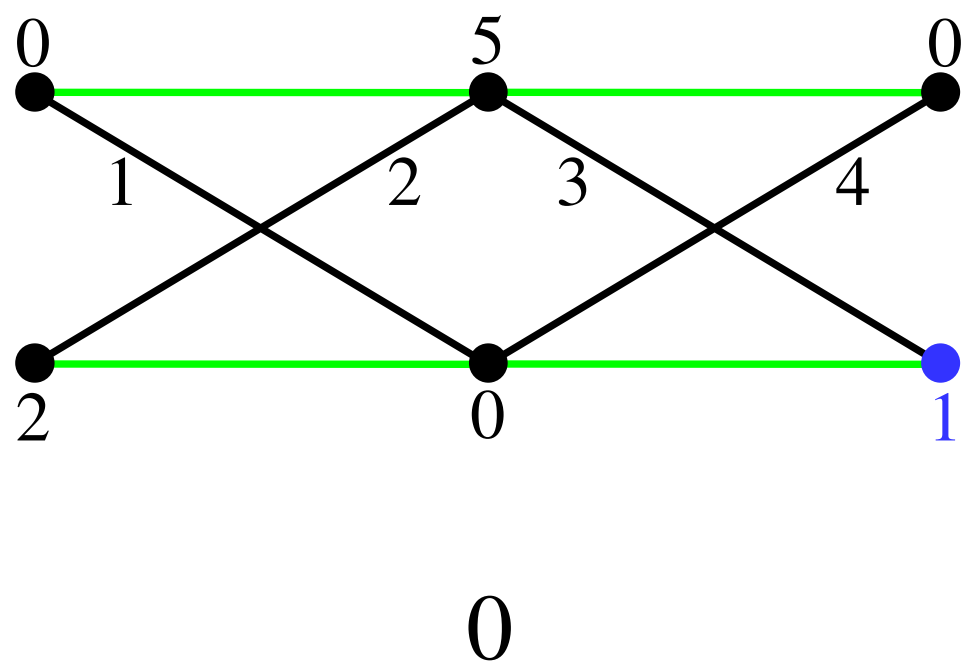

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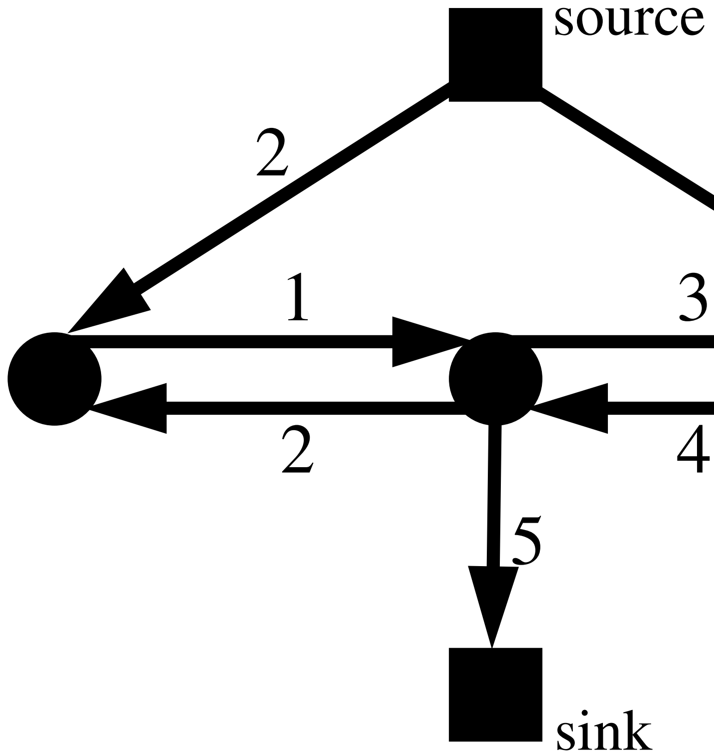
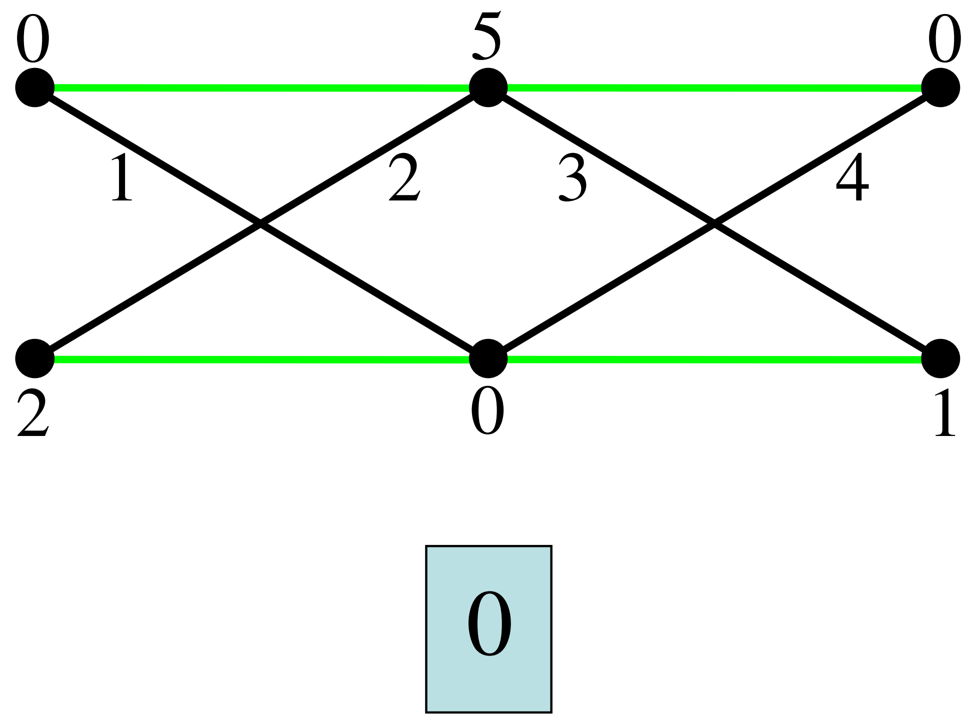
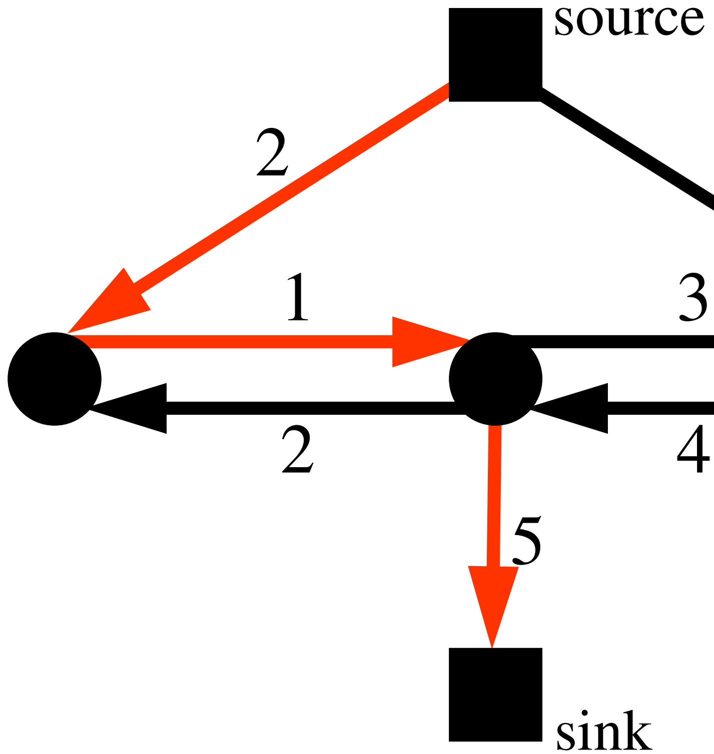
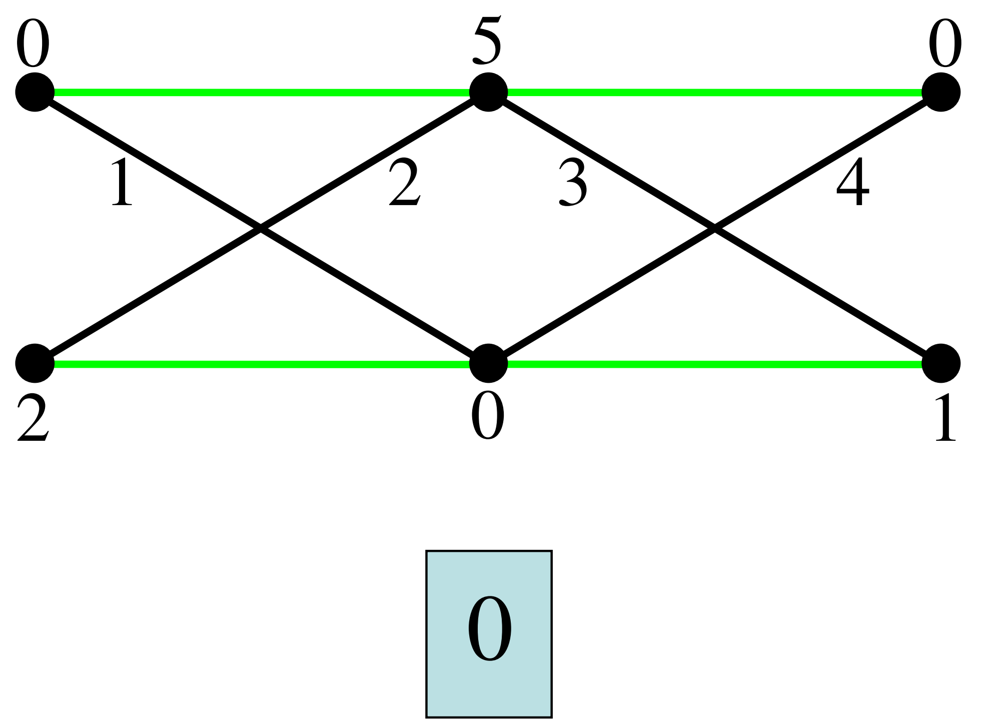




value(flow)=0

Maxflow algorithm
and reparameterisation





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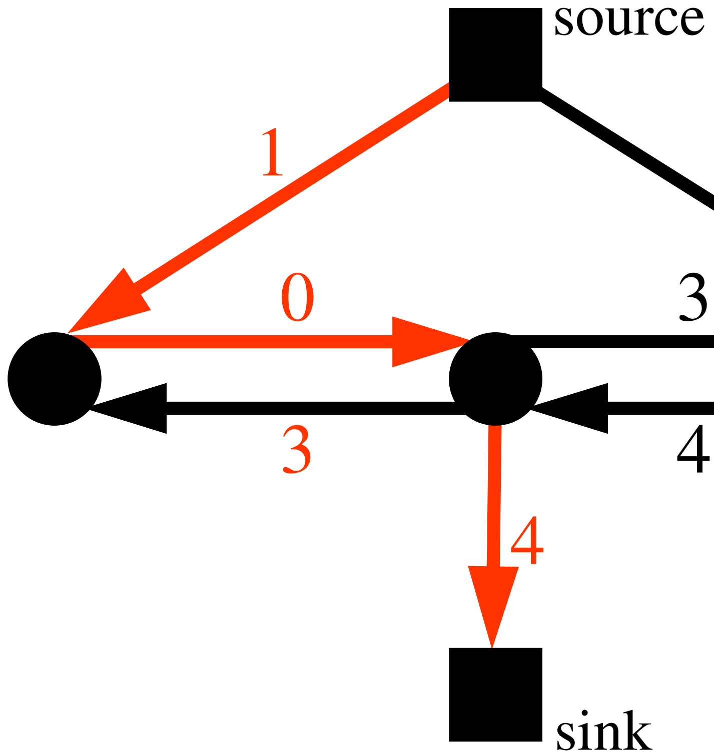
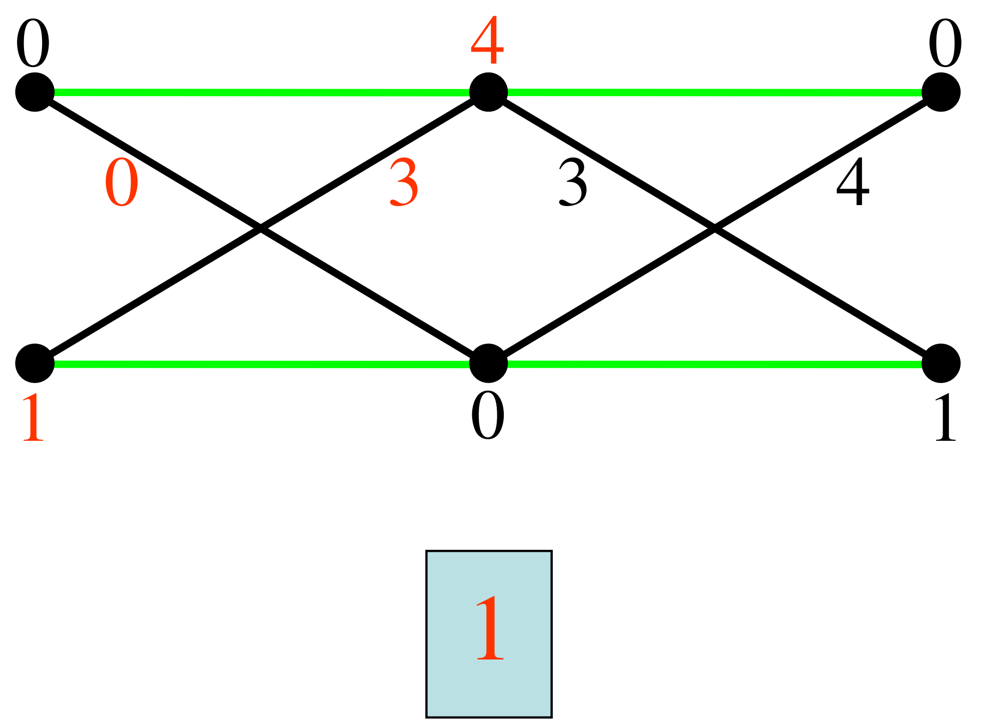
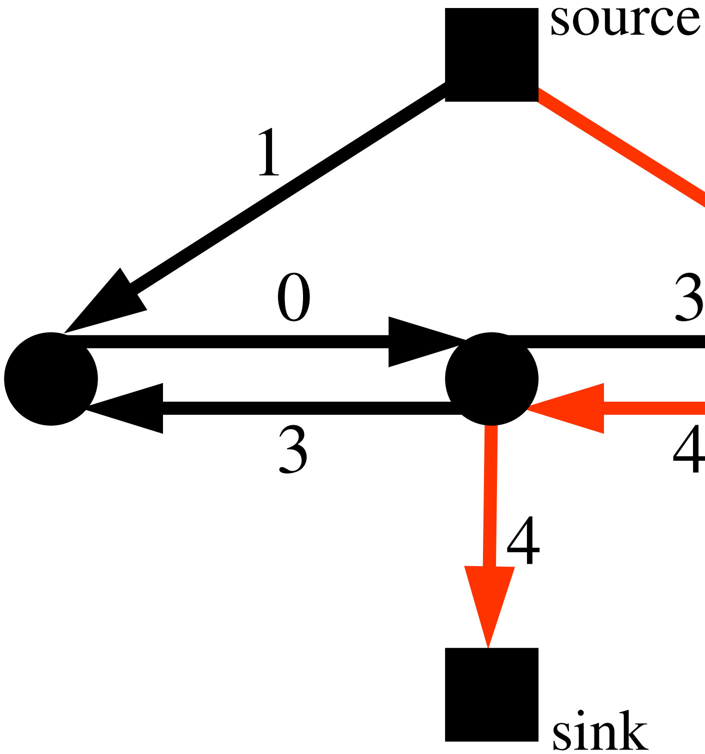
Maxflow algorithm and reparameterisation

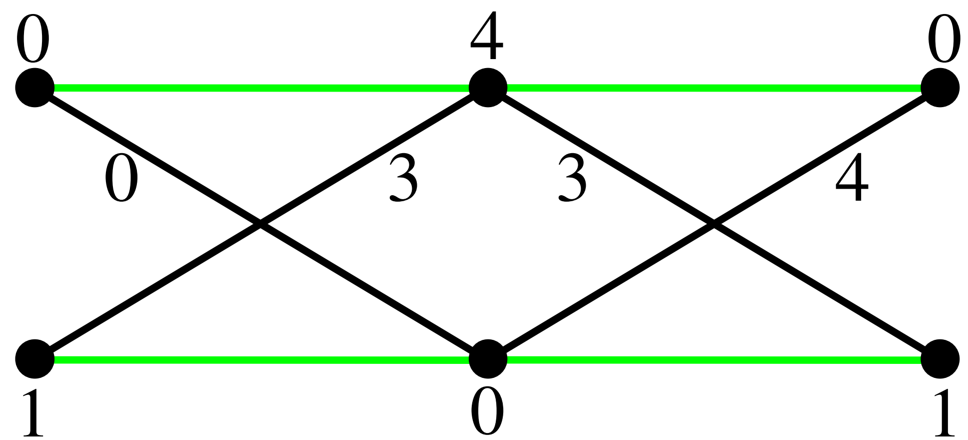




value(flow)=1

Maxflow algorithm and reparameterisation

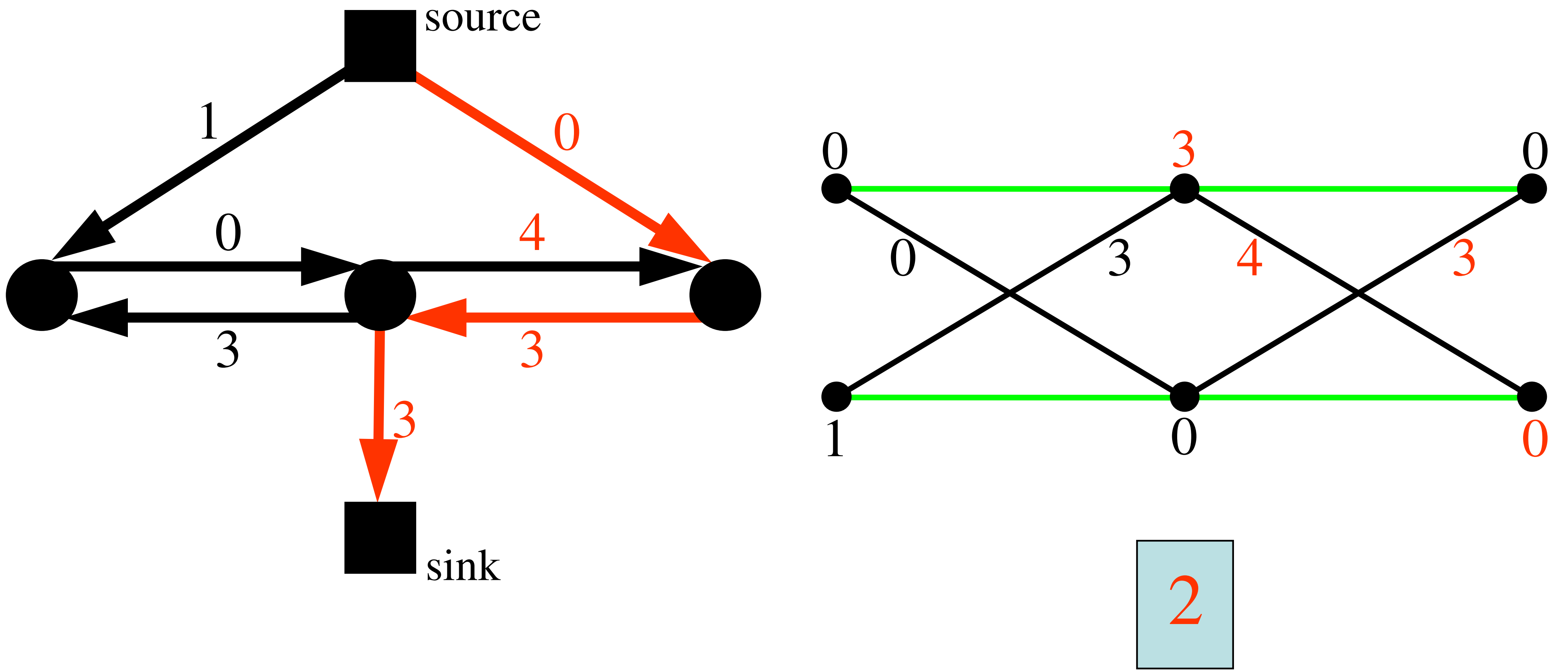




1

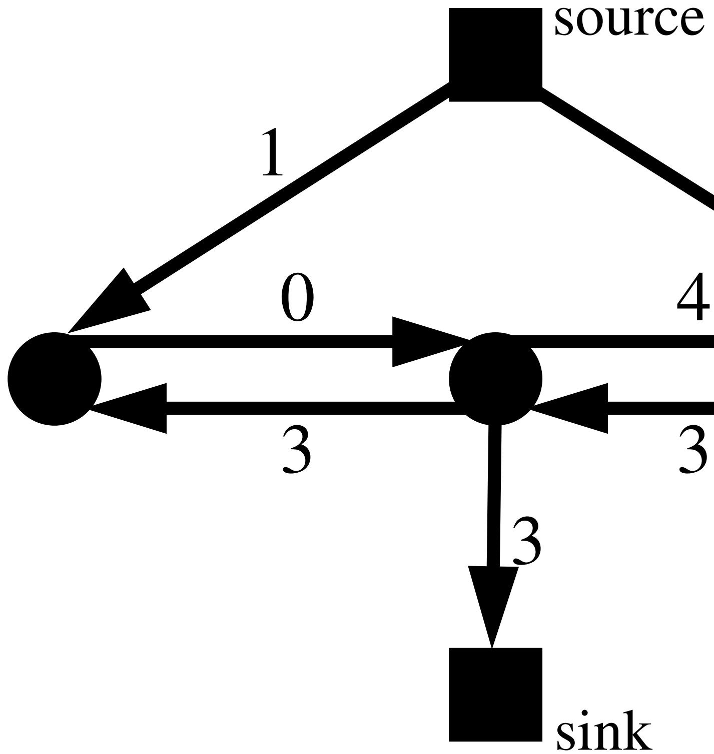
value(flow)=1

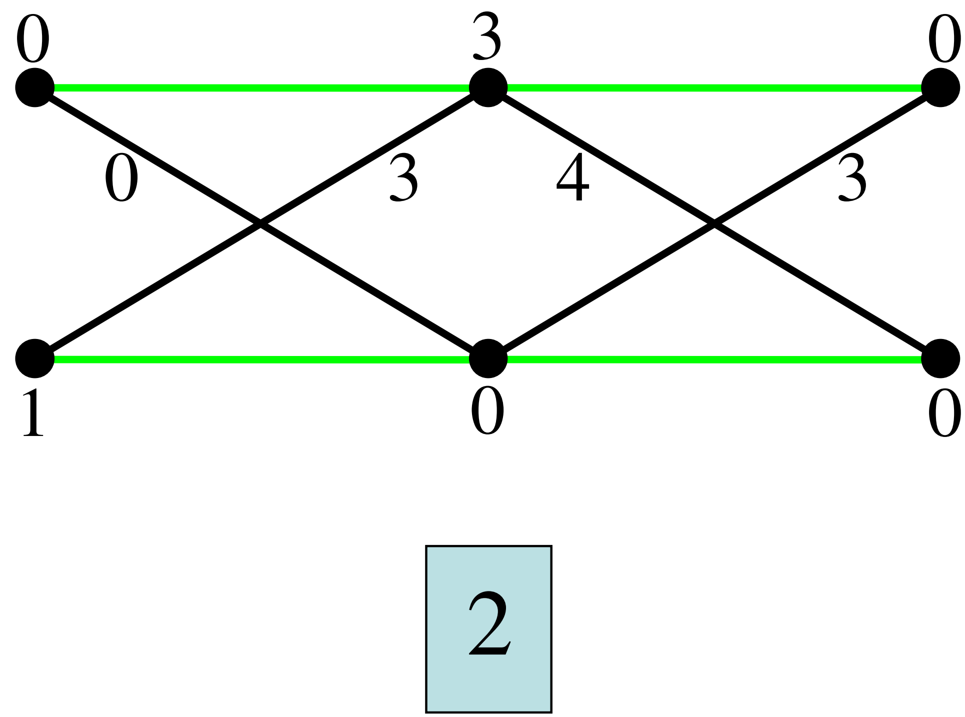
Maxflow algorithm and reparameterisation



value(flow)=2

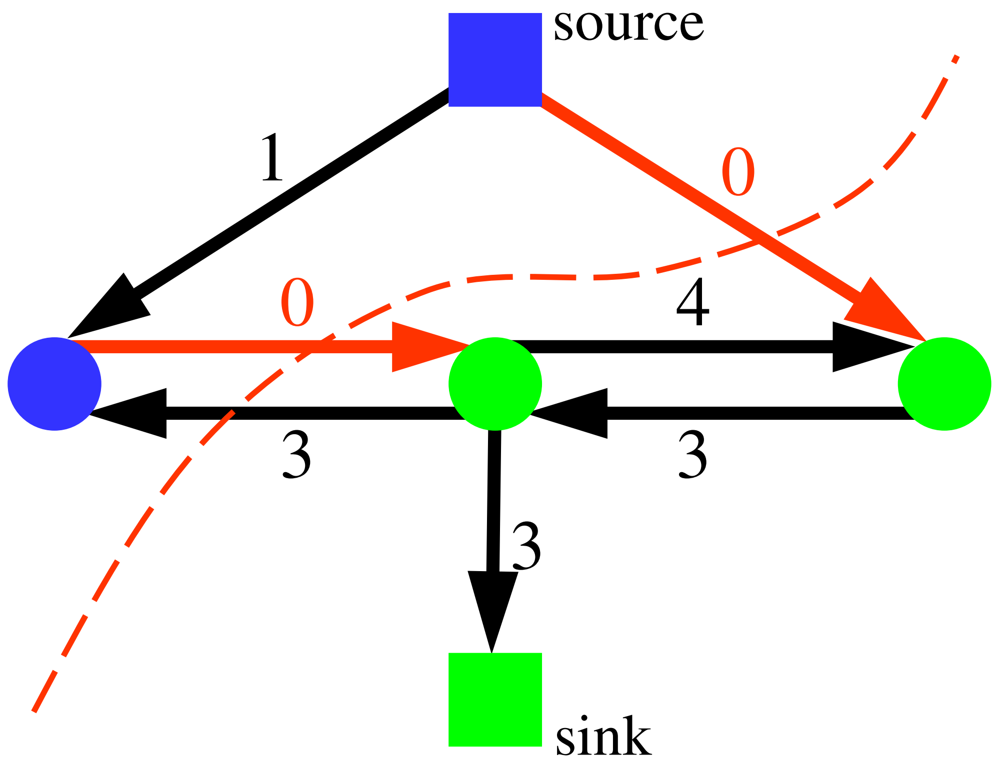
Maxflow algorithm and reparameterisation



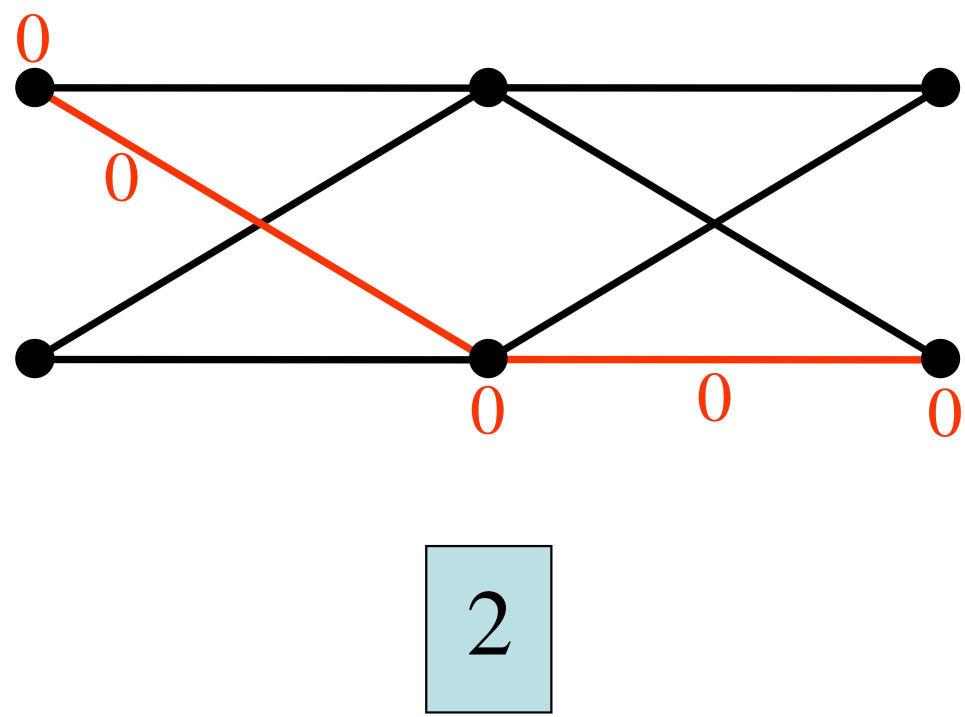


value(flow)=2

Maxflow algorithm and reparameterisation



value(flow)=2



minimum of the energy:

$$\mathbf{x} = (0,1,1)$$

Binary variables,
non-submodular functions

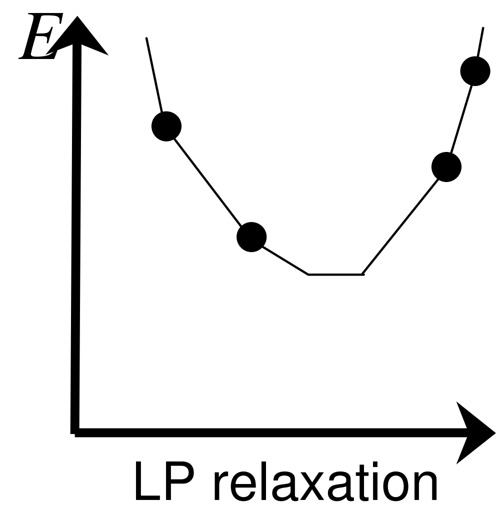
Arbitrary functions of binary variables

$$E(\mathbf{x} \mid \theta) = \theta_{const} + \sum_{p} \theta_p(x_p) + \sum_{p,q} \theta_{pq}(x_p, x_q)$$

maximize

non-negative

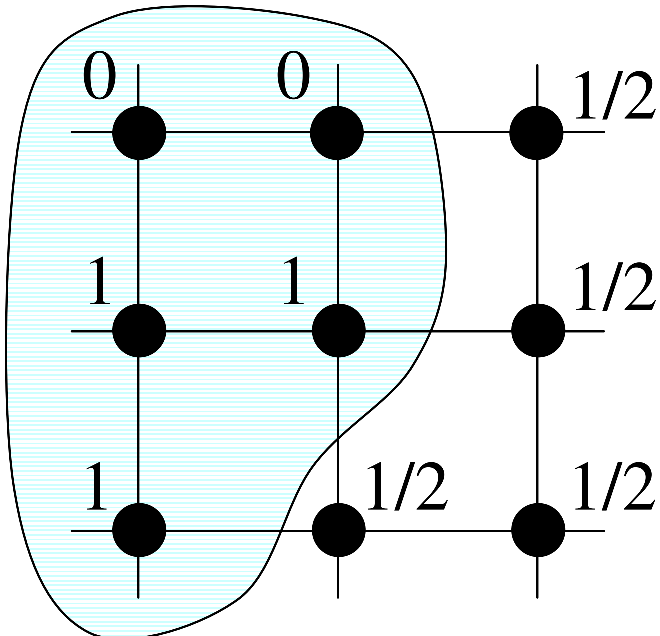
- Can be solved via maxflow [Hammer,Hansen,Simeone'84][Boros,Hammer,Sun'91]
  - Specially constructed graph



- Gives solution to LP relaxation: for each node

$$x_p \in \{0, 1/2, 1\}$$

Arbitrary functions of binary variables



Part of optimal solution
[Hammer, Hansen, Simeone'84]

Graph construction - Main idea

$$E(\{x_p\}) = \sum E_p(x_p)$$ — unary

$$+ \sum E_{pq}(x_p, x_q)$$ — pairwise submodular

$$+ \sum \tilde{E}_{pq}(x_p, x_q)$$ — pairwise non-submodular

- Double # of variables: $x_p \rightarrow x_p, x_{\bar{p}}$
  - Ideally, $x_{\bar{p}} = 1 - x_p$
- Write *E* as a function of both old and new variables
  - New function is submodular!

Graph construction - Main idea

$$E(\{x_p\}) = \sum E_p(x_p) + \sum E_{pq}(x_p, x_q) + \sum \tilde{E}_{pq}(x_p, x_q)$$

$$E(\{x_p\},\{x_{\overline{p}}\}) = \sum \frac{E_p(x_p) + E_p(1 - x_{\overline{p}})}{2} + \sum \frac{E_{pq}(x_p, x_p) + E_p(1 - x_{\overline{p}}, 1 - x_{\overline{q}})}{2} + \sum \frac{\tilde{E}_{pq}(x_p, 1 - x_{\overline{q}}) + \tilde{E}_p(1 - x_{\overline{p}}, x_q)}{2}$$

- Double # of variables: $x_p \rightarrow x_p, x_{\overline{p}}$
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Graph construction - Main idea

$$E(\{x_p\}) = \sum E_p(x_p)$$

$$E(\{x_p\},\{x_{\bar{p}}\}) = \sum \frac{E_p(x_p) + E_p(1 - x_{\bar{p}})}{2}$$

- Double # of variables: $x_p \rightarrow x_p, x_{\bar{p}}$
  - Ideally, $x_{\bar{p}} = 1 - x_p$
- Write *E* as a function of both old and new variables
  - New function is submodular!

Graph construction - Main idea

$$+\sum E_{pq}(x_p, x_q)$$

$$+\sum \frac{E_{pq}(x_p, x_p) + E_p(1 - x_{\overline{p}}, 1 - x_{\overline{q}})}{2}$$

- Double # of variables: $x_p \rightarrow x_p, x_{\overline{p}}$
  - Ideally, $x_{\overline{p}} = 1 - x_p$
- Write *E* as a function of both old and new variables
  - New function is submodular!

Graph construction - Main idea

$$+\sum \tilde{E}_{pq}(x_p, x_q)$$

$$+\sum \frac{\tilde{E}_{pq}(x_p, 1-x_{\bar{q}}) + \tilde{E}_p(1-x_{\bar{p}}, x_q)}{2}$$

- Double # of variables: $x_p \rightarrow x_p, x_{\bar{p}}$
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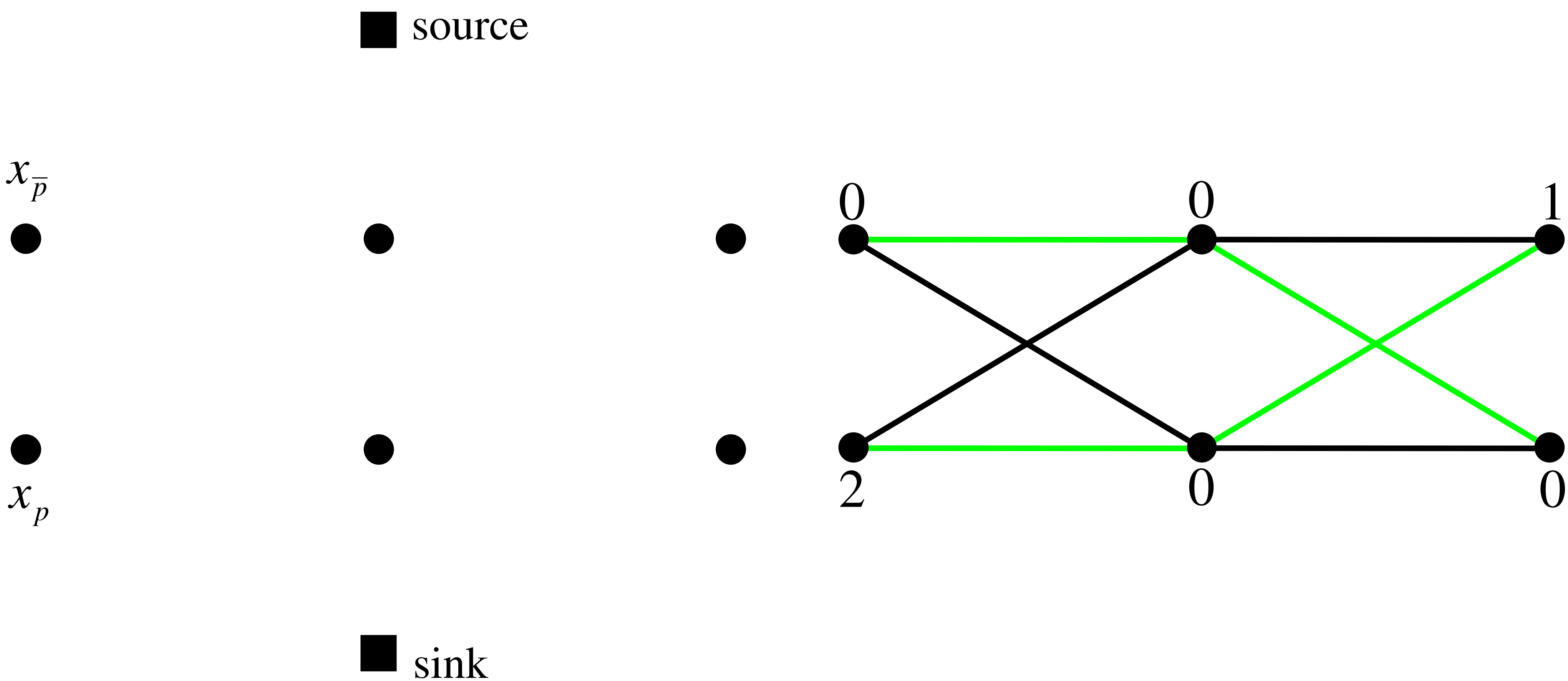
Graph construction - Main idea

$$E(\{x_p\}) = \sum E_p(x_p) + \sum E_{pq}(x_p, x_q) + \sum \tilde{E}_{pq}(x_p, x_q)$$

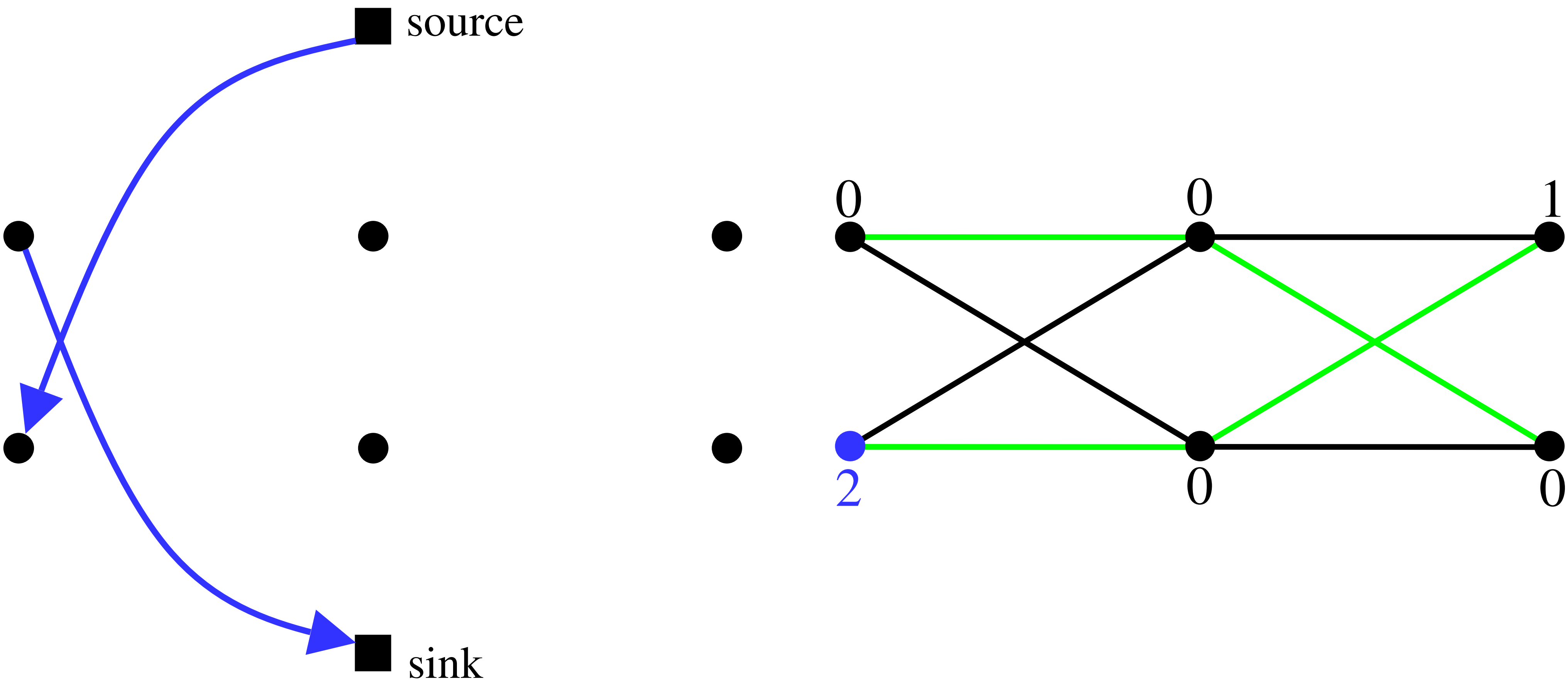
$$E(\{x_p\},\{x_{\bar{p}}\}) = \sum \frac{E_p(x_p) + E_p(1 - x_{\bar{p}})}{2} + \sum \frac{E_{pq}(x_p, x_p) + E_p(1 - x_{\bar{p}}, 1 - x_{\bar{q}})}{2} + \sum \frac{\tilde{E}_{pq}(x_p, 1 - x_{\bar{q}}) + \tilde{E}_p(1 - x_{\bar{p}}, x_q)}{2}$$

- Minimise new function $E(\{x_p\},\{x_{\bar{p}}\})$
  - Without constraint $x_{\bar{p}} = 1 - x_p$

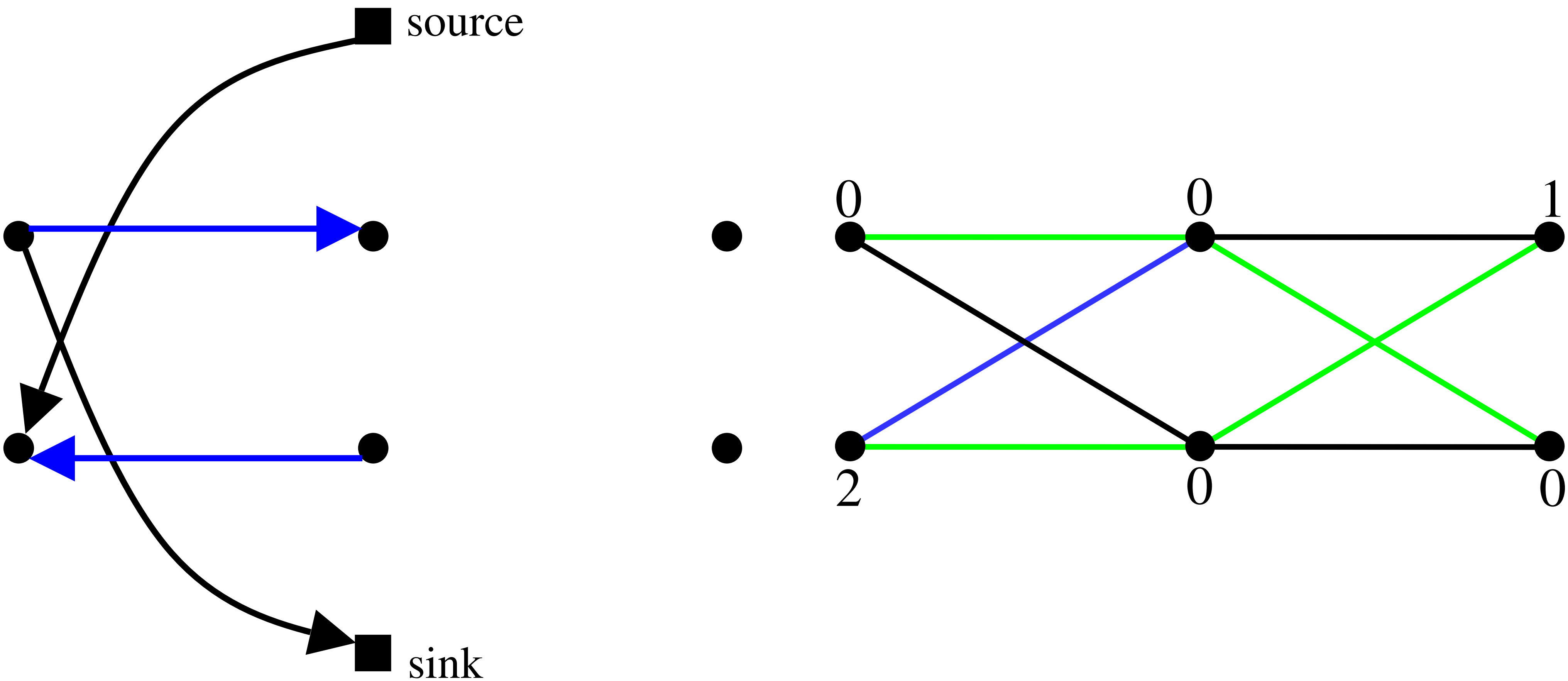
Graph construction



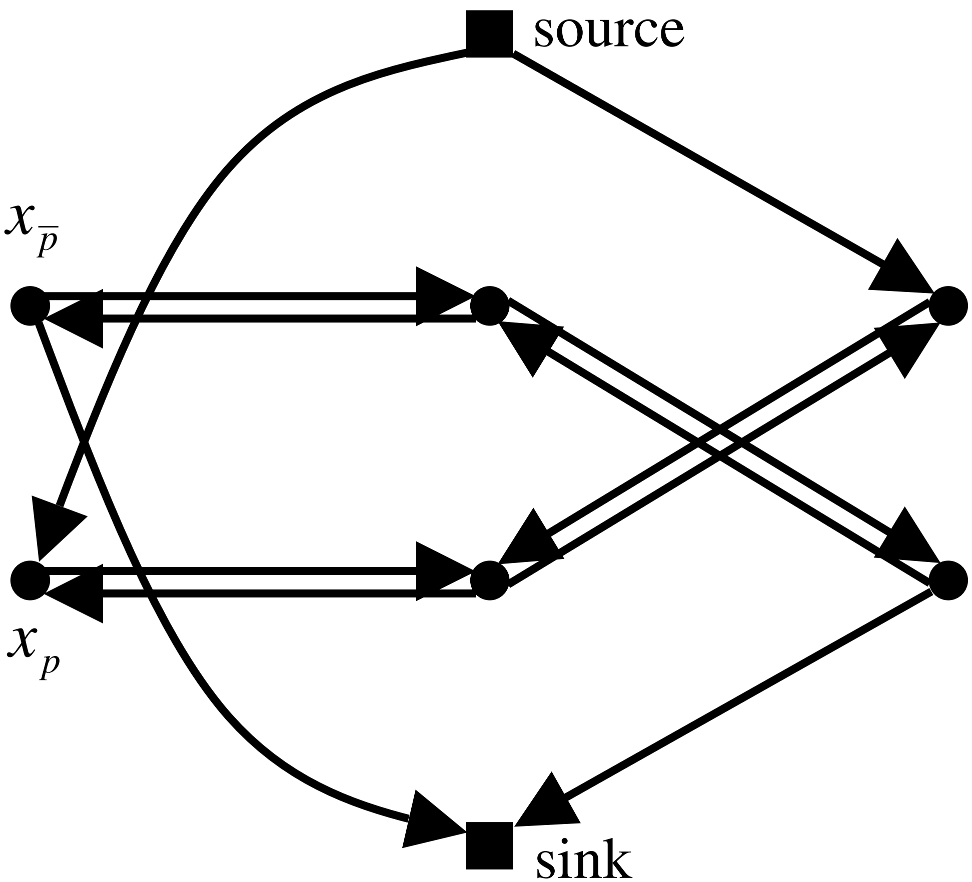
Graph construction

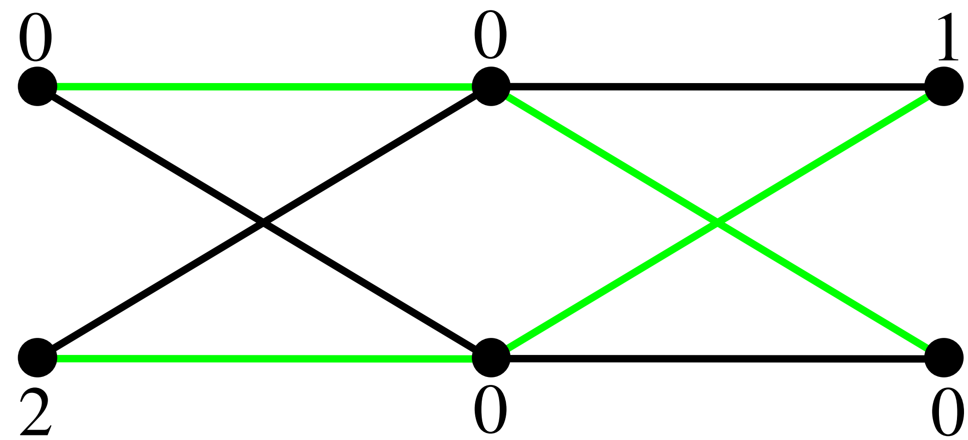


Graph construction



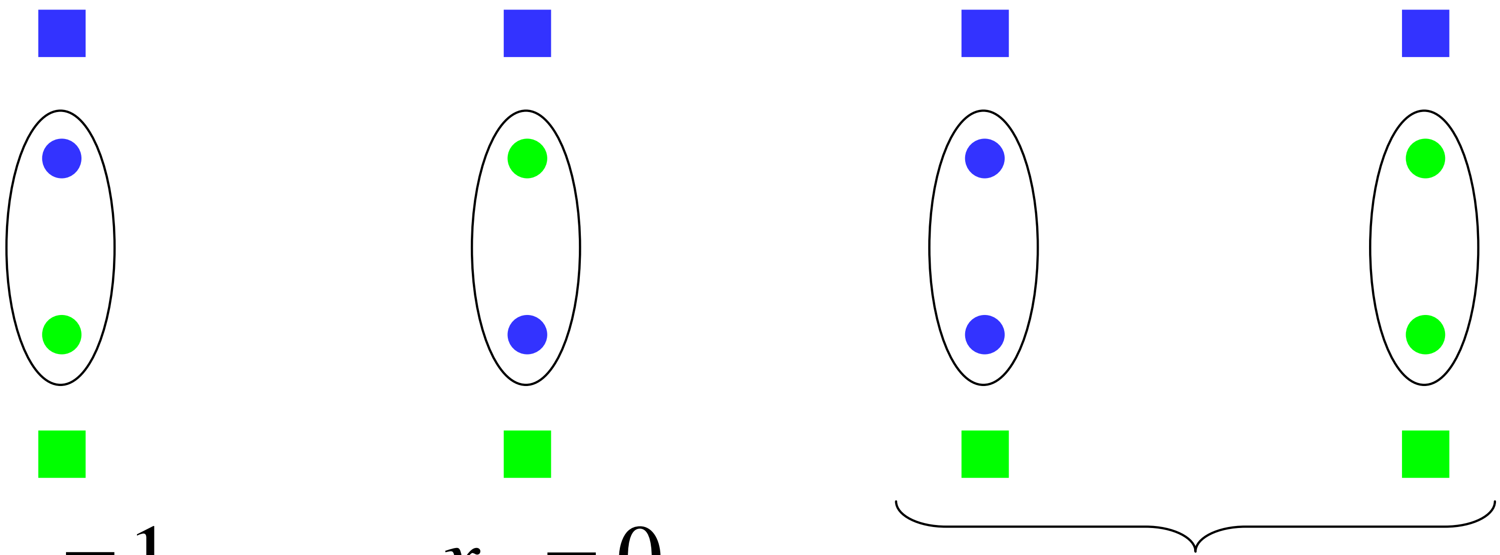
Graph construction





Assigning labels

- Assign labels based on minimum cut in auxiliary graph



$x_p = 1$

$x_{\bar{p}} = 0$

$x_p = 0$

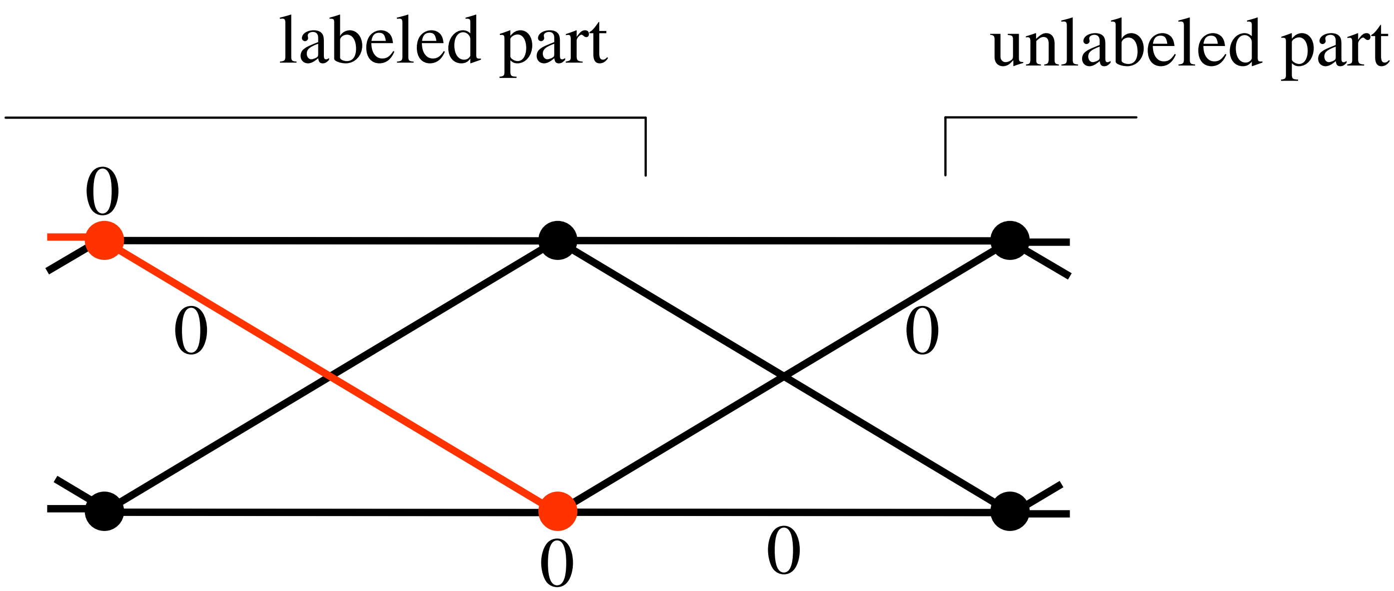
$x_{\bar{p}} = 1$

node p is unlabeled

- To maximize # of labeled nodes, choose a particular minimum cut

Optimality

Theorem [Hammer,Hansen,Simeone'84].
Labeling **x** is part of optimal labeling $\mathbf{x}^*$.



$x_p = 0$ $x_p = 1$ $x_p = ?$

Part B: Lower bound via convex combination of trees

($\Rightarrow$ tree-reweighted message passing)

Convex combination of trees [Wainwright, Jaakkola, Willsky '02]

- Goal: compute minimum of the energy for θ:

$$\Phi(\theta) = \min_{\mathbf{x}} E(\mathbf{x} \mid \theta)$$

- In general, intractable!

- Obtaining lower bound:
  - Split θ into several components: $\theta = \theta^1 + \theta^2 + \ldots$
  - Compute minimum for each component:

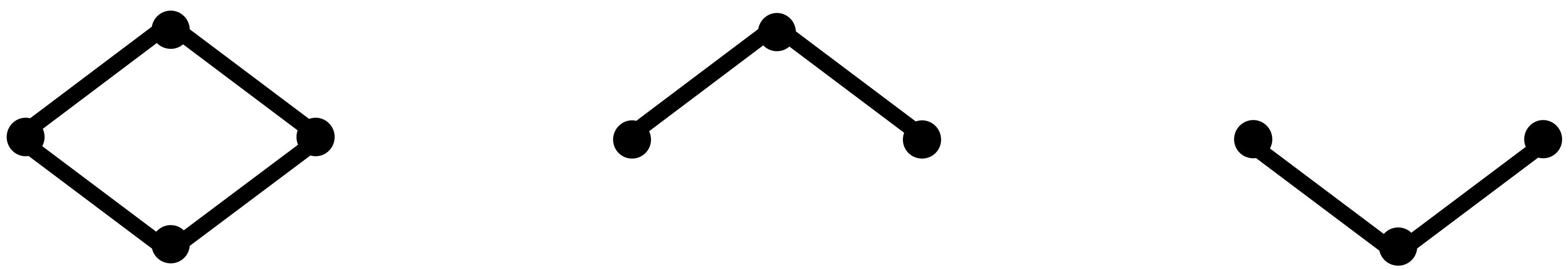
$$\Phi(\theta^i) = \min_{\mathbf{x}} E(\mathbf{x} \mid \theta^i)$$

  - Combine $\Phi(\theta^1)$, $\Phi(\theta^2)$, ... to get a bound on $\Phi(\theta)$

- Use trees!

Convex combination of trees (cont'd)

graph — tree T — tree T'



$$\theta \equiv \frac{1}{2}\theta^{T} + \frac{1}{2}\theta^{T'}$$

$$\Phi(\theta) \geq \underbrace{\frac{1}{2}\Phi(\theta^{T}) + \frac{1}{2}\Phi(\theta^{T'})}_{\text{lower bound on the energy}}$$

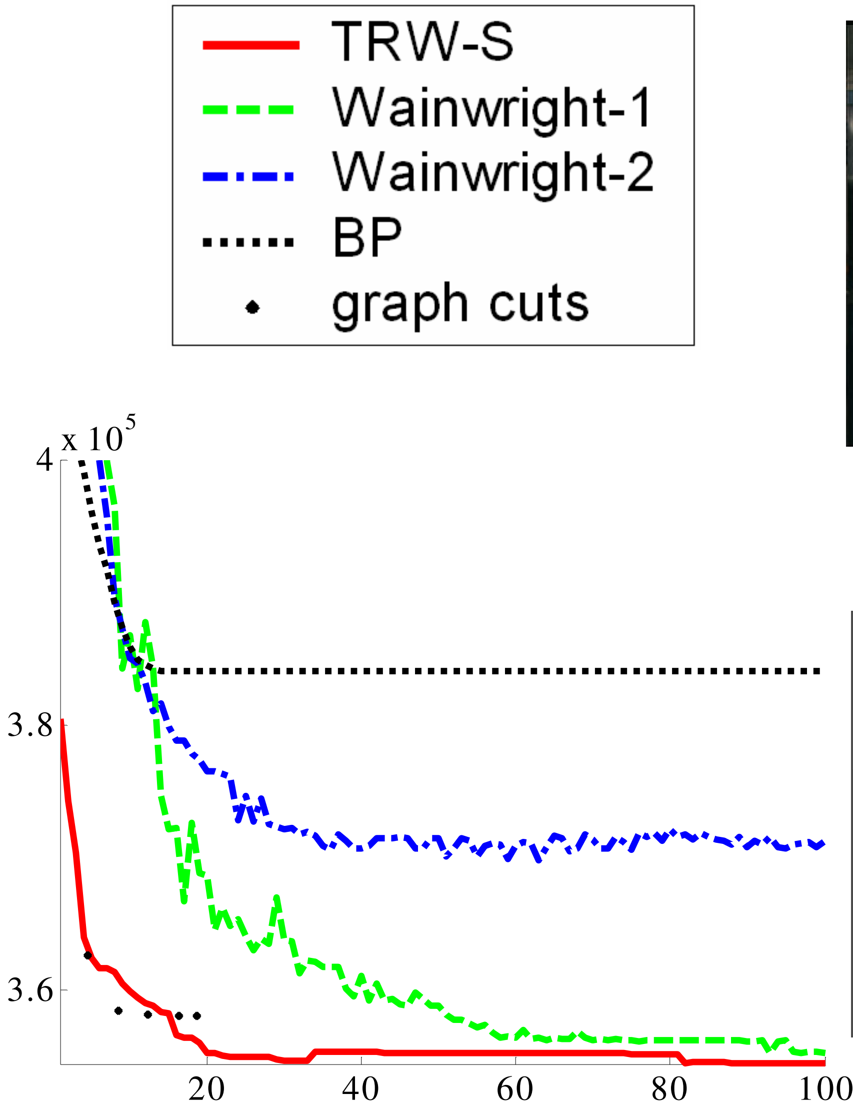
maximize

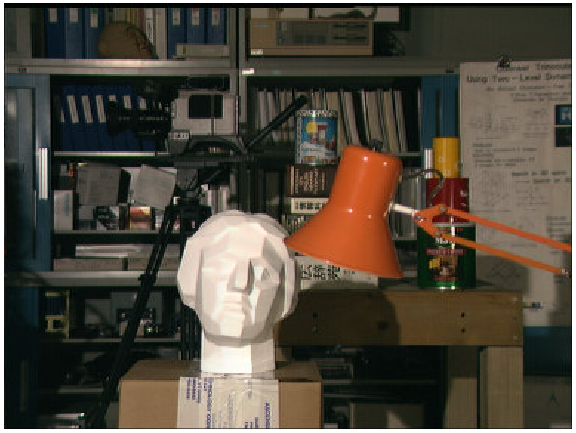
lower bound on the energy

TRW algorithms

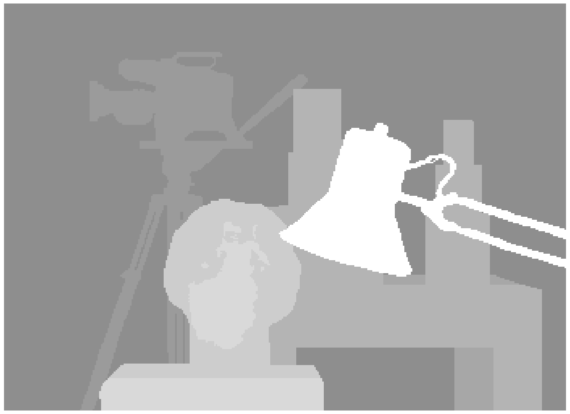
- Goal: find reparameterisation maximizing lower bound
- Apply sequence of different reparameterisation operations:
  - Node averaging
  - Ordinary BP on trees
- Order of operations?
  - Affects performance dramatically
- Algorithms:
  - [Wainwright *et al*. '02]: parallel schedule
    - May not converge
  - [Kolmogorov'05]: specific sequential schedule
    - Lower bound does not decrease, convergence guarantees
    - Needs half the memory

Experimental results: stereo

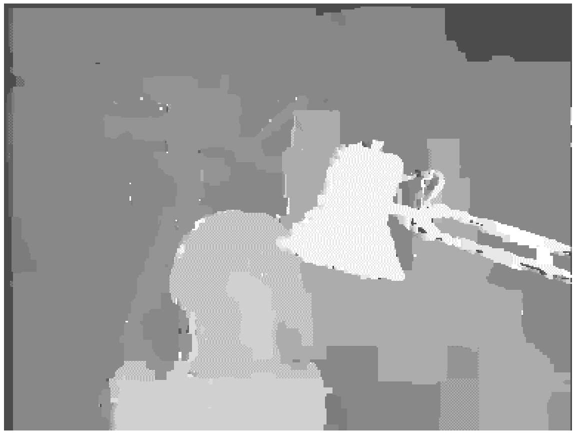




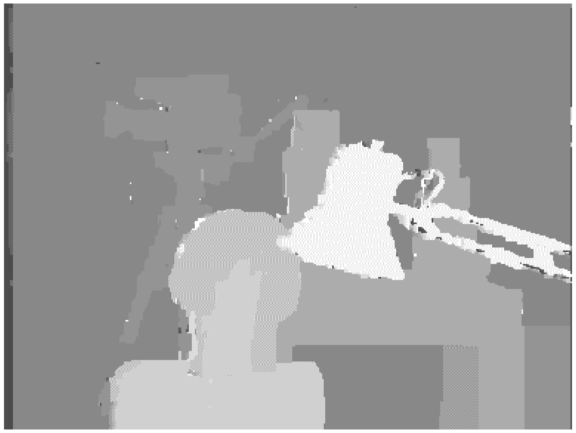
left image



ground truth



BP



TRW-S

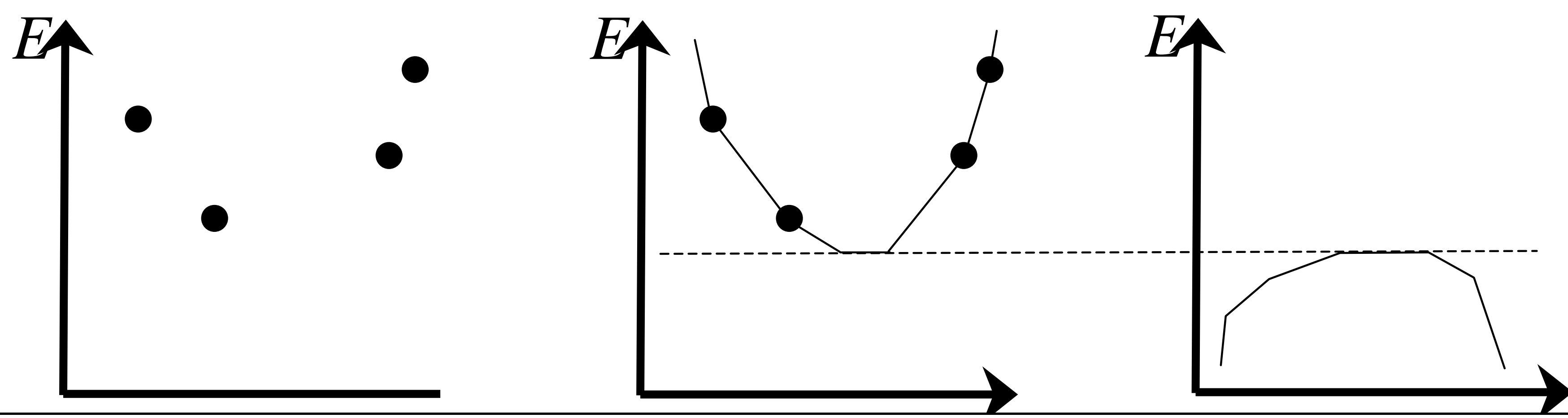
- Global minima for some instances with TRW [Meltzer,Yanover,Weiss'05]

Parts A and B: Summary

- MAP estimation algorithms are based on LP relaxation
  - Maximize lower bound
- Two ways to formulate lower bound
- Via posiforms: leads to maxflow algorithm (for binary variables)
  - Polynomial time solution
  - Submodular functions: global minimum
  - Non-submodular functions: part of optimal solution
- Via convex combination of trees: leads to TRW algorithm
  - Convergence in the limit (for TRW-S)
  - Applicable to arbitrary energy function

Non-binary variables: Other methods for solving LP

- No polynomial-time algorithm (except general purpose LP solvers)
- Iterative methods:
  - [Koval,Schlesinger'76]: *augmenting DAG algorithm*
  - [Kovalevsky,Koval'75, Flach'98] (unpublished): *max-sum diffusion*
    - See tech. report [Werner'05]
    - Not guaranteed to solve LP (only *arc constistent* solution) – same as TRW
- Special case: *submodular functions*
  - LP has integer optimal solution [Schlesinger,Flach'00]
  - Reduction to maxflow [Ishikawa'03, D.Schlesinger'05]



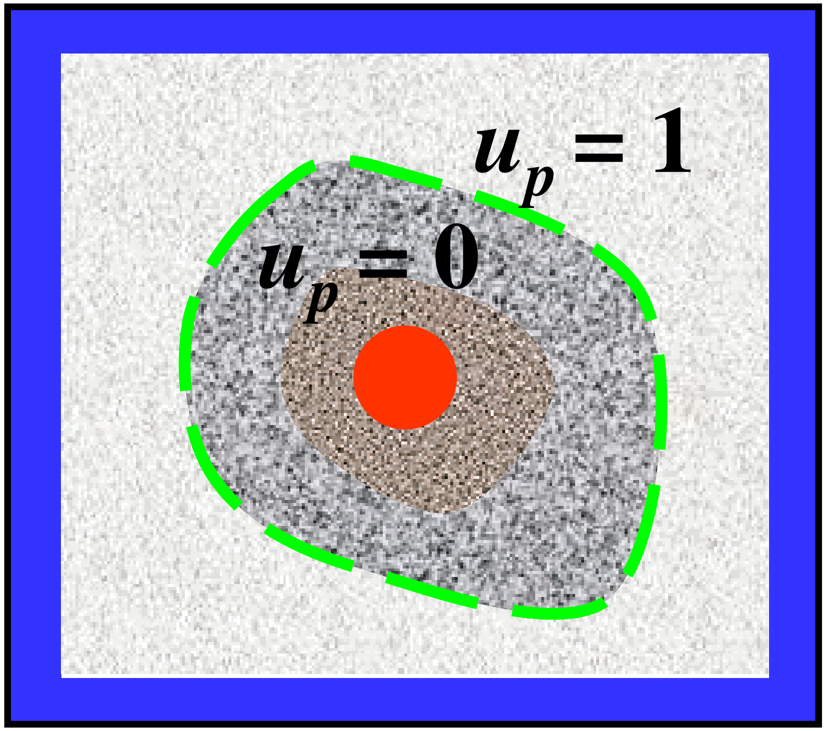
Continuous mincut/maxflow

Continuous mincut/maxflow

- Primal problem:

$$\int_C g(C(s))\, ds \rightarrow \min$$

subject to $\begin{cases} s \text{ inside } C \\ t \text{ outside } C \end{cases}$



Alternatively:

$$\int |\nabla u|_g \rightarrow \min$$

subject to $\begin{cases} u_p = 0,\ p \in s \\ u_p = 1,\ p \in t \end{cases}$

<u>total variation</u>

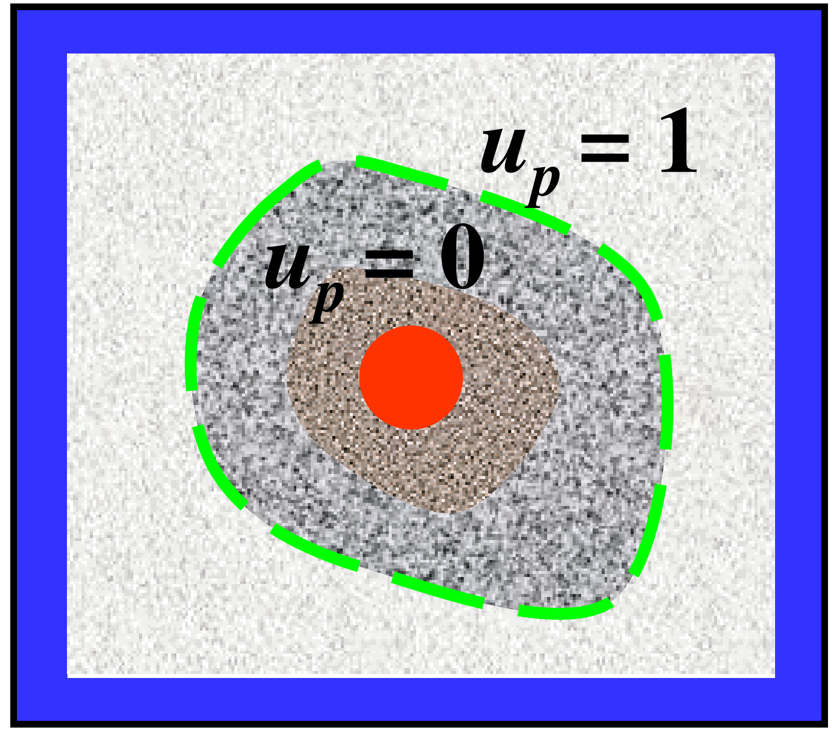
[Rudin,Osher,Fatemi'92] :
image restoration
[Amar,Belletini'94]:
definition for arbitrary metric

Continuous mincut/maxflow

- Primal problem:

$$\int_C g(C(s))\, ds \to \min$$

subject to $\begin{cases} s \text{ inside } C \\ t \text{ outside } C \end{cases}$



Alternatively:

$$\int |\nabla u|_g \to \min$$

subject to $\begin{cases} u_p = 0, p \in s \\ u_p = 1, p \in t \end{cases}$

- $u_p \in [0,1]$ – *fractional segmentations*
- Convex problem
- Integer optimal solution

Continuous mincut/maxflow

- Dual problem:

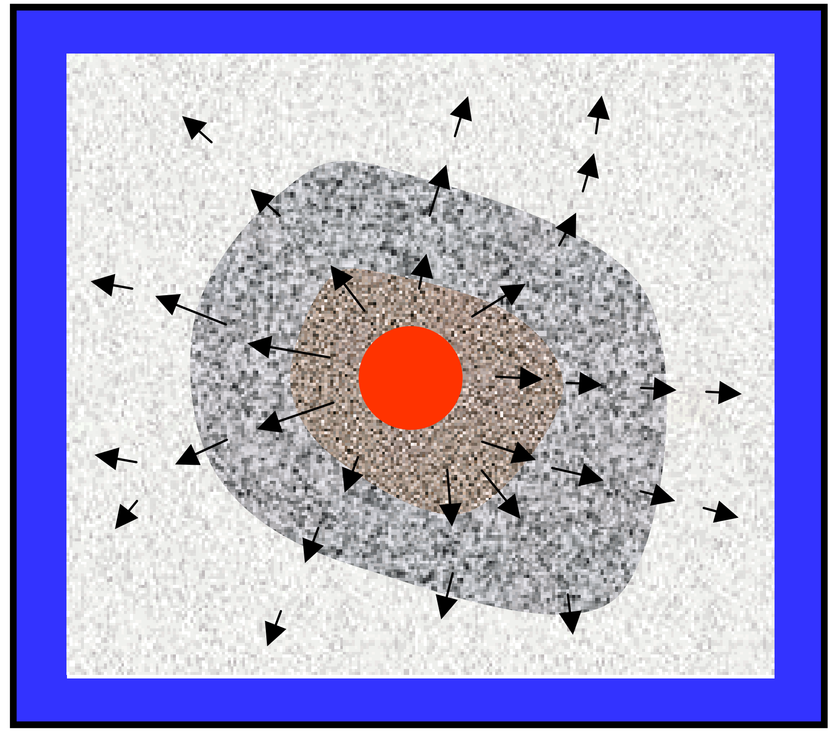
$$\int_{S} (\operatorname{div} \vec{f}_p)\, da \rightarrow \max$$

subject to

$$|\vec{f}_p| \leq g$$ *(capacity constraint)*

$$\operatorname{div} \vec{f}_p = 0$$ *(flow conservation)*

for $p \notin s,t$



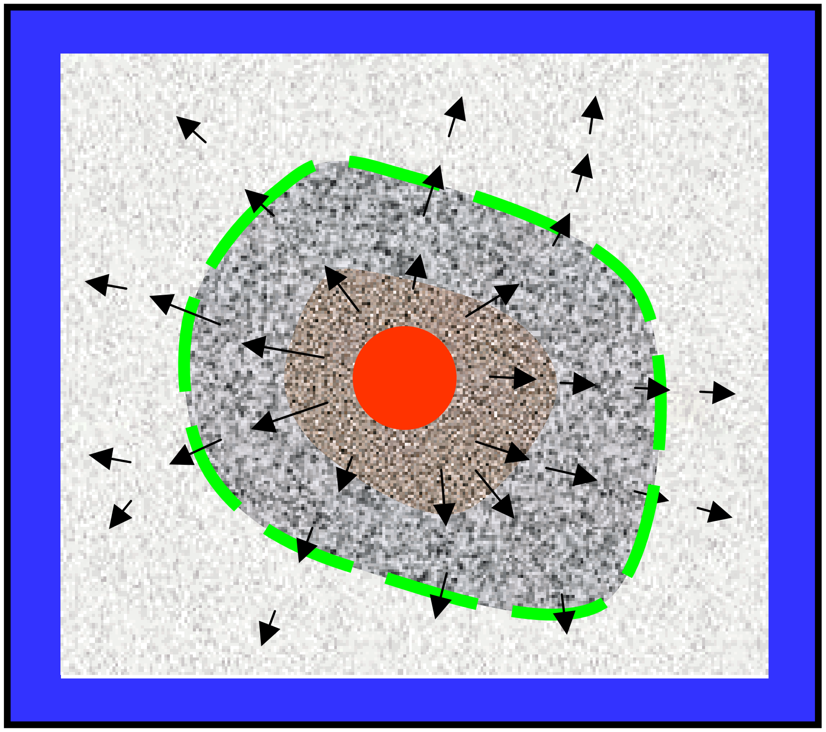
Reparameterisation

- Any flow with

$$\text{div}\,\vec{f}_p = 0 \quad \text{for} \quad p \notin s,t$$

defines reparameterisation

(*by the divergence theorem*):



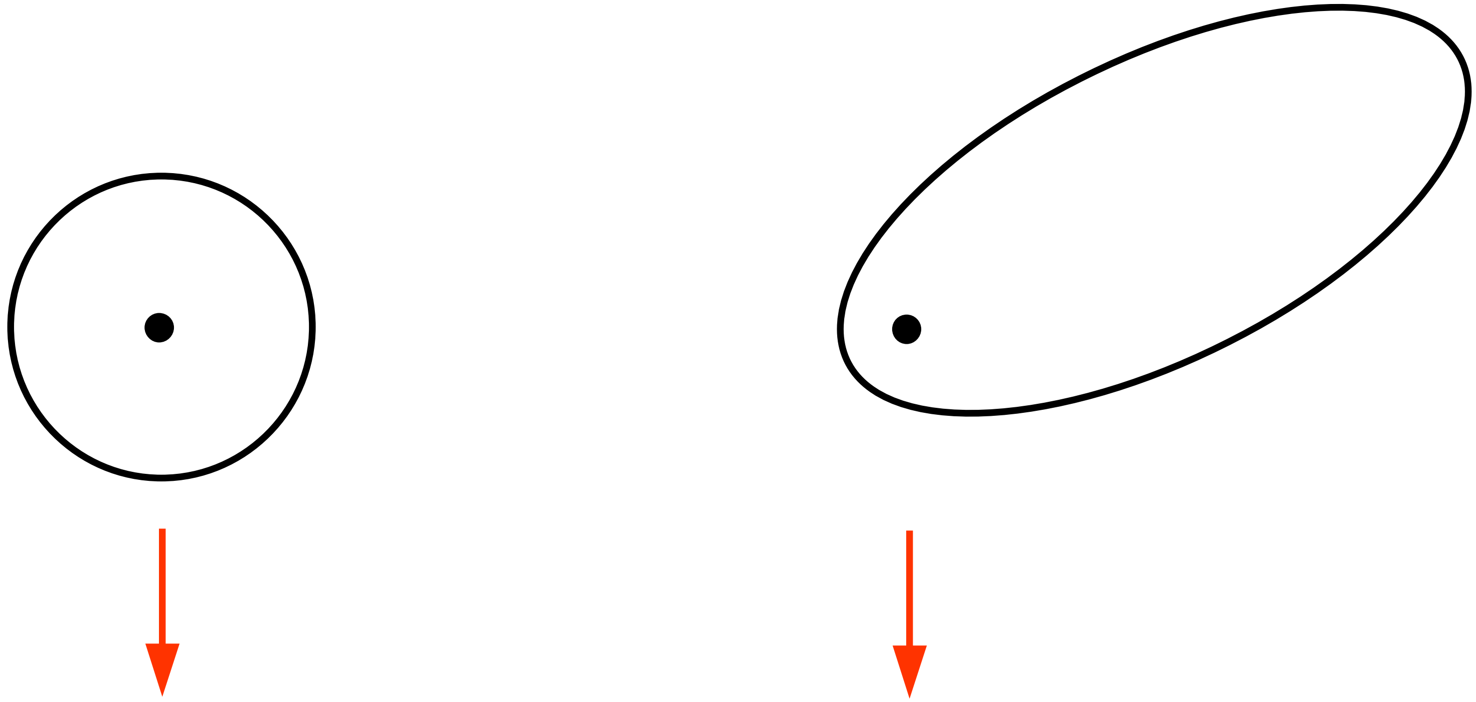
$$E(C) \equiv \int_C g\,ds = const + \int_C (g - \vec{f} \cdot \vec{N})\,ds$$

where

$$const = \int_s (\text{div}\,\vec{f})\,ds$$

Reparameterisation

distance maps



$$E(C) \equiv \int_C g \, ds = const + \int_C (g - \vec{f} \cdot \vec{N}) \, ds$$

lower bound on $E(C)$

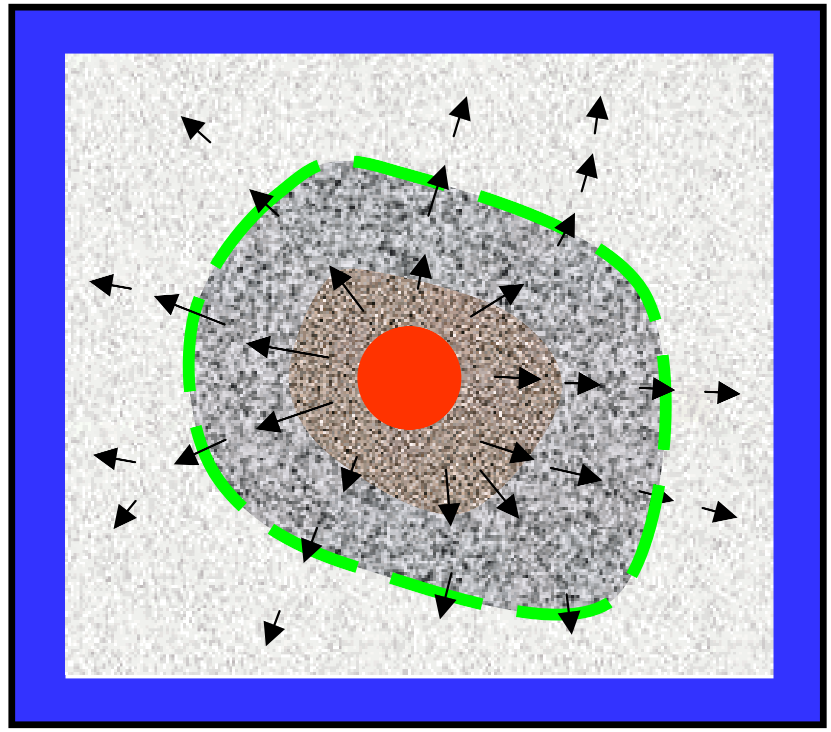
$| \vec{f} | \leq g \;\Rightarrow$ non-negative

Reparameterisation

Suppose flow saturates cut C^*

$(\vec{f}_p = g_p \vec{N} \quad \text{for } p \in C^*)$:

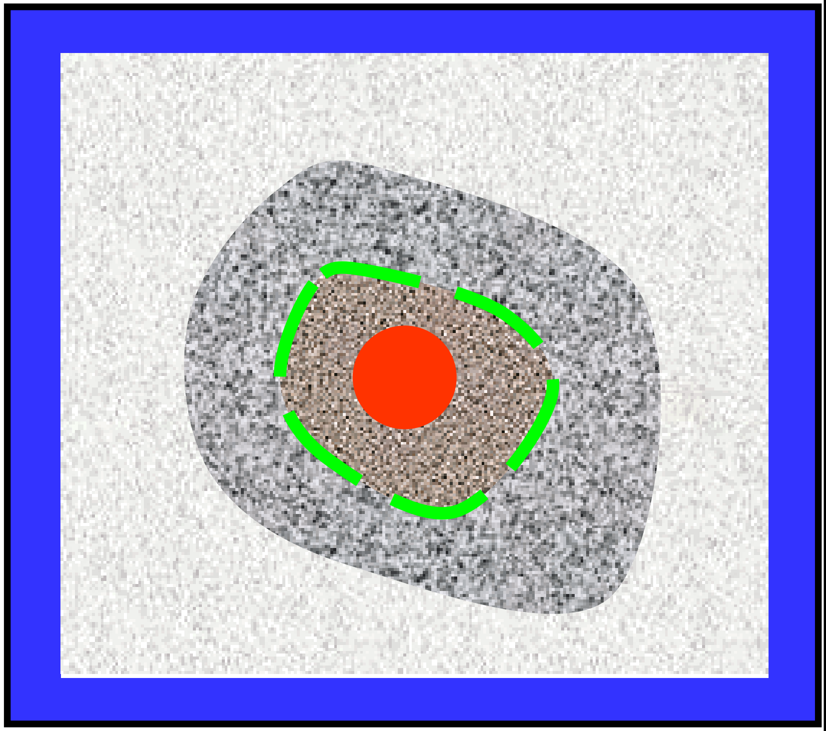
$\Rightarrow \; C^* = \text{minimum cut}$



$$E(C) \equiv \int_C g\, ds = const + \underbrace{\int_C (g - \vec{f} \cdot \vec{N})\, ds}_{\text{zero for } C^*}$$

Global vs. local optimisation algorithms: Summary

- Geodesic active contours
  - Variational approach (e.g. level sets)
    - Gradient descent in the *space of contours*
    - Local minimum
    - Non-convex formulation
  - Graph cuts (e.g. geo-cuts)
    - Extended space (fractional segmentations)
    - Convex formulation
    - Integer solution (for submodular functions)



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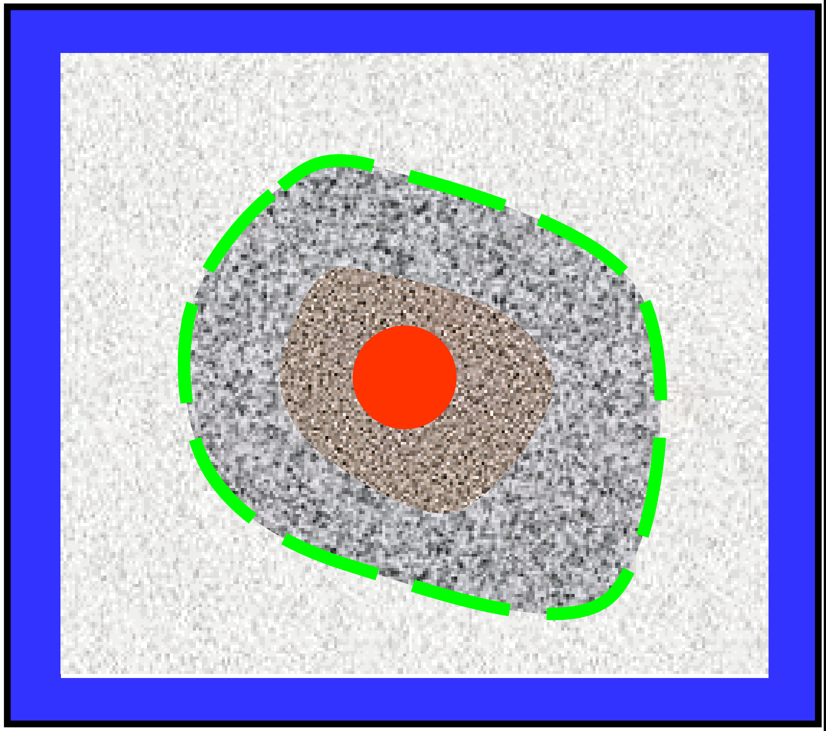


Other relaxations/extensions

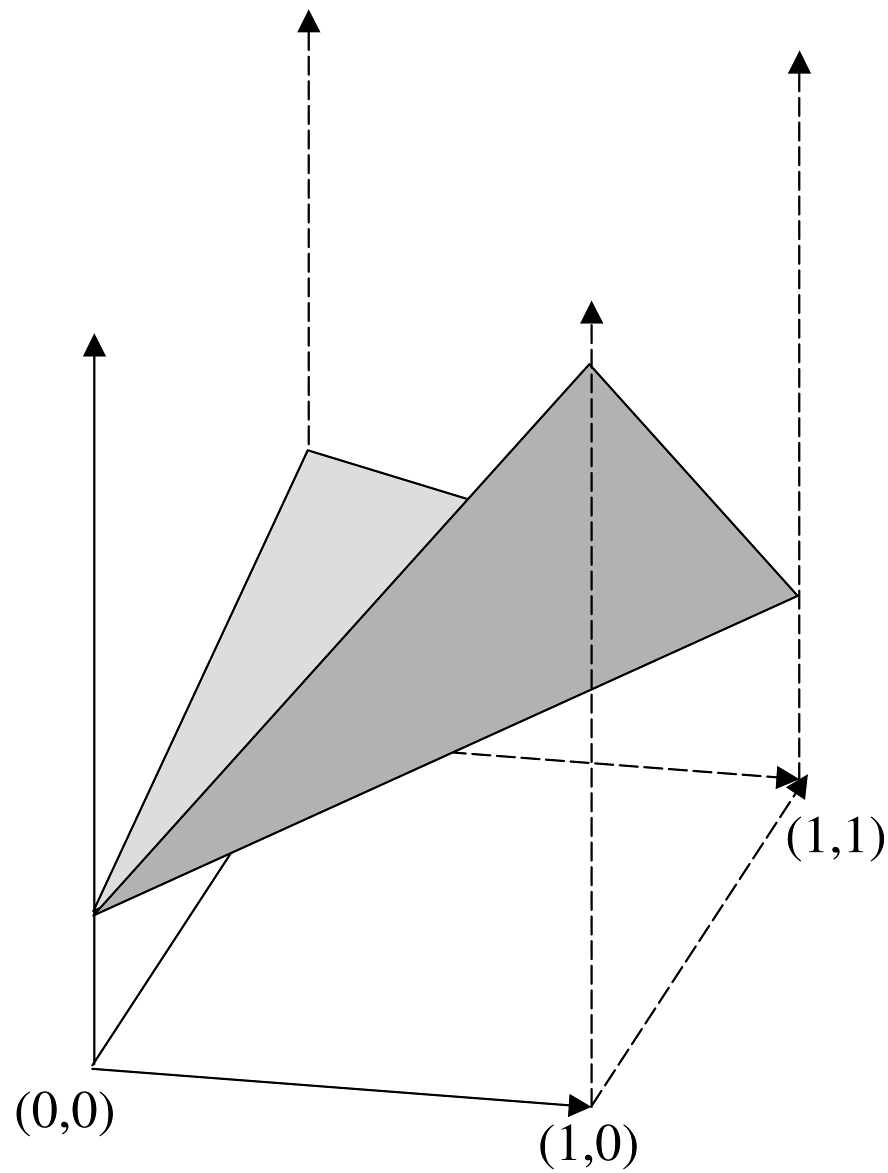
- Energy $E(\mathbf{x})$ defined for integer configurations $(x_p \in \{0,1\})$
- How to define for fractional configurations $(x_p \in [0,1])$?

Other relaxations/extensions

- LP relaxation [Schlesinger'76,Koster et al.'98,Chekuri et al.'00, Wainwright et al.'03]
  - Defined for multi-valued variables
  - Convex
  - E is submodular $\Rightarrow$ integer solution
- Lovász extension [Lovász'83]
  - Defined for binary variables
  - Always integer solution
  - E is submodular $\Leftrightarrow$ extension is convex
  - "Submodularity" – discrete analogue of convexity
- Sherali-Adams relaxation, semi-definite relaxation, SOCP relaxation, ...

LP relaxation and Lovász extension

Submodular function: $E(0,0) + E(1,1) \leq E(0,1) + E(1,0)$



LP relaxation and Lovász extension

Non-submodular function: $E(0,0) + E(1,1) \geq E(0,1) + E(1,0)$